

# ON THE SEMIAMPLENESS OF THE POSITIVE PART OF CKM ZARISKI DECOMPOSITIONS

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**ABSTRACT.** We investigate the semiampleness of the positive part of CKM Zariski decompositions of big divisors on LC pairs whose difference with the log canonical divisor is nef. We give some conjectures that would generalize an important theorem of Kawamata and we prove them in low dimension. Moreover we give a relative version of DLT pair and we prove a similar statement in this setting.

## 1. INTRODUCTION

A very important tool to understand the asymptotic behaviour of a line bundle is the Zariski decomposition.

A  $\mathbb{Q}$ -Cartier divisor  $D$  on a projective variety  $X$  admits a  $\mathbb{Q}$ -Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki (or a  $\mathbb{Q}$ -CKM Zariski decomposition)  $D = P + N$  if

- $P$  and  $N$  are  $\mathbb{Q}$ -Cartier divisors;
- $P$  is nef and  $N$  is effective;
- There exists an integer  $k > 0$  such that  $kD$  and  $kP$  are integral divisors and for every  $m \in \mathbb{N}$  the natural map

$$H^0(X, \mathcal{O}_X(kmP)) \rightarrow H^0(X, \mathcal{O}_X(kmD))$$

is bijective.

Every pseudoeffective divisor on a smooth projective surface admits a Zariski decomposition (see [Fuj79]). On the other hand in higher dimension there exist big divisors such that no birational pullbacks admit a Zariski decomposition, as shown by Cutkosky in [Cut86], even if we allow  $P$  and  $N$  to be  $\mathbb{R}$ -divisors (see [Nak04]).

If  $D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition, then, up to pass to a multiple, the graded ring  $R(X, D) := \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mD))$  is isomorphic to  $R(X, P)$ , so that in order to check its finite generation it suffices to study the finite generation of the ring associated to the nef divisor  $P$ . This implies that the semiampleness of  $P$ , the so-called positive part of the Zariski decomposition of  $D$ , is a sufficient condition for the finite generation of  $R(X, D)$ . See [Mor87, (9.11)] for the importance of the Zariski decomposition in the context of the *Abundance Conjecture*.

Our starting point is the main theorem of the paper [Kaw87]. (A trivial generalization of) Kawamata's theorem says that:

**Theorem 1.1** (Kawamata). *Let  $X$  be a normal projective variety and let  $\Delta$  be a Weil effective  $\mathbb{Q}$ -divisor such that  $(X, \Delta)$  is a KLT pair. If  $D$  is a  $\mathbb{Q}$ -Cartier divisor such that*

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- A:**  $D$  is big;  
**B:**  $aD - (K_X + \Delta)$  is nef for some rational number  $a \geq 0$ ;  
**C:**  $D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $D = P + N$ ;

then the positive part  $P$  is semiample, so that  $R(X, D)$  is finitely generated.

In this paper we want to consider the case when  $(X, \Delta)$  is an LC pair.

The same theorem is no longer true in general in this case, as shown in section 8, so that we have to assume more. For this reason we use the notion of logbig divisors, introduced by Miles Reid.

A big  $\mathbb{Q}$ -Cartier divisor is logbig for an LC pair if its restriction to every LC center of the pair is still big (see definition 4.1).

We state the following conjecture:

**Conjecture 1.** *Let  $(X, \Delta)$  be an LC pair, with  $\Delta$  effective.*

*If  $D$  is a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$  which satisfies **A**, **B** and **C**, and  $P$  is logbig for the pair  $(X, \Delta)$ , then  $P$  is semiample.*

In the case  $N = 0$ , a very similar result was stated by Miles Reid in [Rei93] and was proved by Florin Ambro in the more general setting of quasi-log varieties in [Amb01, theorem 7.2] (see also [Fuj09a, theorem 4.4]). Moreover, when  $X$  is smooth and  $\Delta$  and  $N$  are SNCS divisors Conjecture 1 is true and follows from [Fuj07b, theorem 5.1]. In fact the saturation condition discussed by Fujino corresponds to the properties of the positive part of the Zariski decomposition.

Note that a sufficient condition for the positive part  $P$  to be logbig for the pair  $(X, \Delta)$  is that the augmented base locus  $\mathbb{B}_+(D)$  does not contain any LC center of the pair. Hence Conjecture 1 would imply the following:

**Conjecture 2.** *Let  $(X, \Delta)$  be an LC pair, with  $\Delta$  effective.*

*If  $D$  is a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$  which satisfies **A**, **B** and **C**, and  $\mathbb{B}_+(D)$  does not contain any LC center of the pair  $(X, \Delta)$ , then  $P$  is semiample.*

On the other hand, note that, for a divisor  $D$ , the existence of a Zariski decomposition is a very strong property in general, while it is more likely that a birational pullback of  $D$  admits one. In other words we want to replace hypothesis **C** with the following:

- C<sub>f</sub>:** There exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D) = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition.

We can thus generalize Conjecture 1 and Conjecture 2 as follows (the “b” stands for “birational”):

**Conjecture 1b.** *Let  $(X, \Delta)$  be an LC pair, with  $\Delta$  effective.*

*Let  $D$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$  satisfying **A**, **B** and **C<sub>f</sub>**, for some  $f : Z \rightarrow X$ . If  $P$  is logbig for the pair  $(Z, \Delta_Z)$ , where  $\Delta_Z$  is a  $\mathbb{Q}$ -divisor on  $Z$  such that  $K_Z + \Delta_Z = f^*(K_X + \Delta)$ , then  $P$  is semiample.*

**Conjecture 2b.** *Let  $(X, \Delta)$  be an LC pair, with  $\Delta$  effective.*

*Let  $D$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$  satisfying **A**, **B** and **C<sub>f</sub>**, for some  $f : Z \rightarrow X$ . If  $\mathbb{B}_+(f^*D)$  does not contain any LC center of the pair  $(Z, \Delta_Z)$ , where  $\Delta_Z$  is a  $\mathbb{Q}$ -divisor on  $Z$  such that  $K_Z + \Delta_Z = f^*(K_X + \Delta)$ , then  $P$  is semiample.*

In this paper we easily deduce from [Fuj07b, theorem 5.1] that Conjecture 1b holds if a suitable pair on  $Z$  is DLT (see theorem 4.2), which in particular implies that

Conjecture 1 holds if  $(X, \Delta)$  is DLT. In section 5 we give sufficient conditions on the Zariski-decomposed divisor and on the geometry of the LC centers of the pair in order to have the semiampleness (see 5.3 and 5.5). As a corollary we prove the following:

**Theorem 1.2** (see corollary 5.6). *Conjecture 1b holds if  $\dim X \leq 3$ .*

Also we prove that Conjecture 2 holds if we assume some standard very strong conjectures concerning the Minimal Model Program, namely the abundance conjecture (for semi log canonical pairs) and the existence of minimal models (in lower dimension) for DLT pairs of log-general type (see theorem 5.7). As a corollary we get that

**Theorem 1.3** (see corollary 5.8). *Conjecture 2 holds if  $\dim X \leq 4$ .*

In section 6 we consider a relative version of DLT pair:

**Definition 1.4** (see definition 6.1). Let  $(X, \Delta)$  be a pair, with  $\Delta = \sum a_i D_i$ , where the  $D_i$ 's are distinct prime divisors and  $a_i \in \mathbb{Q}$  for every  $i$ . Let  $D \in \text{Div}_{\mathbb{Q}}(X)$ . We define the *non-simple normal crossing locus* of  $(X, \Delta)$  as the closed set

$$\text{NSNC}(\Delta) = X \setminus U,$$

where  $U$  is the biggest open subset of  $X$  such that  $\Delta|_U$  has simple normal crossing support.

We say that  $(X, \Delta)$  is a *D-DLT pair* if

- (1)  $V \not\subseteq \text{Sing}(X) \cup \text{NSNC}(\Delta)$  for every LC center  $V$  of the pair  $(X, \Delta)$  such that  $V \cap \mathbb{B}(D) \neq \emptyset$ ;
- (2)  $a_i \leq 1$  for every  $i$  such that  $D_i \cap \mathbb{B}(D) \neq \emptyset$ ;

Note that a D-DLT pair is not necessarily LC. As a particular case of theorem 6.9 we prove the following:

**Theorem 1.5** (see corollary 6.10). *Let  $(X, \Delta)$  be a pair such that  $\Delta$  is effective. Let  $D$  be a  $\mathbb{Q}$ -Cartier divisor on  $X$  satisfying **A**, **B** and **C<sub>f</sub>**, where  $f : Z \rightarrow X$ , and let  $\Delta_Z$  be a  $\mathbb{Q}$ -divisor on  $Z$  such that  $K_Z + \Delta_Z = f^*(K_X + \Delta)$ . If  $(Z, \Delta_Z)$  is  $f^*(D)$ -DLT and  $\mathbb{B}_+(f^*(D))$  does not contain any LC center of the pair  $(Z, \Delta_Z)$ , then  $P$  is semiample.*

This was inspired by [Amb05, Theorem 2.1].

We say that a pair  $(X, \Delta)$  is weak log Fano if  $-(K_X + \Delta)$  is big and nef. Using the previous theorem we find sufficient conditions for the semiampleness of the log anticanonical divisor of (possibly non-LC) weak log Fano pairs:

**Corollary 1.6** (see corollary 6.12). *Let  $(X, \Delta)$  be a weak log Fano pair. Suppose that*

- $(X, \Delta)$  is a  $-(K_X + \Delta)$ -DLT pair;
- $\mathbb{B}_+(-(K_X + \Delta))$  does not contain any LC center of the pair  $(X, \Delta)$ ;

*then  $-(K_X + \Delta)$  is semiample.*

See [Gon09] and [Gon10] for related results in the LC case.

Note that in many statements we do not need  $a \geq 0$  in hypothesis **B**. Moreover some of our theorems hold under the more usual hypotheses

**B'**:  $aD - (K_X + \Delta)$  big and nef for some  $a \in \mathbb{Q}^+$ ;

**C**:  $D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $D = P + N$ ,

as shown in section 7.

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## 2. PRELIMINARIES

**2.1. Notation and conventions.** We will work over the field of complex number  $\mathbb{C}$ . We denote by  $Div(X)$  the set of Cartier divisors on  $X$  and by  $Div_{\mathbb{Q}}(X)$  the set of divisors such that an integral multiple is Cartier.

A *pair*  $(X, \Delta)$  consists of a normal projective variety  $X$  and a Weil  $\mathbb{Q}$ -divisor  $\Delta$  on  $X$  such that  $K_X + \Delta \in Div_{\mathbb{Q}}(X)$ .

We say that a subvariety  $V \subseteq X$  is a *log canonical center* or a *LC center* of the pair  $(X, \Delta)$  if it is the image, through a proper birational morphism, of a divisor  $E$  over  $X$  such that the discrepancy  $a(E, X, \Delta) \leq -1$ .

We define  $CLC(X, \Delta) = \{\text{LC centers of the pair } (X, \Delta)\}$ .

Given  $S \subseteq X$  a closed subset we define

$$CLC(X, \Delta, S) = \{V \in CLC(X, \Delta) : V \cap S \neq \emptyset\}.$$

We say that the pair  $(X, \Delta)$  is *S-KLT* if  $CLC(X, \Delta, S) = \emptyset$ . We say that  $(X, \Delta)$  is *D-KLT* if  $CLC(X, \Delta, \mathbb{B}(D)) = \emptyset$ .

We refer to [KM00] for the standard definitions about singularities of pairs that we do not give explicitly.

**Definition 2.1.** Let  $X$  be a normal variety and let  $D$  be a Weil  $\mathbb{R}$ -divisor on  $X$ . If we write  $D = \sum d_i D_i$ , where the  $D_i$  are distinct prime divisors, we define

$$D^{\geq 1} = \sum_{d_i \geq 1} d_i D_i, \quad D^{=1} = \sum_{d_i=1} D_i.$$

**2.2. Standard log-resolutions.** Let  $X$  be a normal projective variety and let  $D$  be a reduced Weil divisor on  $X$ .

A *standard log-resolution* of the pair  $(X, D)$  is a log-resolution  $f$  of the pair  $(X, D)$  such that

- $f$  is a composition of blowings-up of smooth subvarieties of codimension greater than 1 up to isomorphisms;
- $f|_{f^{-1}(U)}$  is an isomorphism, where  $U = X \setminus (NSNC(D) \cup Sing(X))$ .

If  $X$  is smooth and  $\mathcal{I} \subseteq \mathcal{O}_X$  is a non zero ideal sheaf, then a *standard log-resolution* of the ideal sheaf  $\mathcal{I}$  is a log-resolution  $g$  of  $\mathcal{I}$  such that  $g$  is a composition of blowings-up of smooth subvarieties of codimension greater than 1 contained in  $\text{Cosupp}(\mathcal{I})$  up to isomorphisms.

In particular  $g|_{g^{-1}(X \setminus \text{Cosupp}(\mathcal{I}))}$  is an isomorphism.

**Remark 2.2.** Given a normal projective variety  $X$  and a reduced Weil divisor  $D$  on  $X$ , there exists a standard log-resolution of the pair  $(X, D)$  (this follows, for example, by [Laz04, theorem 4.1.3] and [Fuj07a, theorem 3.5.1]).

If  $Y$  is a smooth projective variety and  $\mathcal{I} \subseteq \mathcal{O}_Y$  is a non zero ideal sheaf, then there exists a standard log-resolution of  $\mathcal{I}$  (see for example [Kol05, theorem 35]).

**Lemma 2.3.** *Let  $(X, \Delta)$  be a LC pair. Let  $L \in \text{Div}_{\mathbb{Q}}(X)$  be big and nef and such that  $\mathbb{B}_+(L)$  does not contain any LC center of the pair  $(X, \Delta)$ .*

*Then there exists an effective Cartier divisor  $\Gamma$  on  $X$ , not containing any LC center of  $(X, \Delta)$  in its support, and a rational number  $\lambda_0 > 0$  such that  $Bs(|\Gamma|) = \mathbb{B}_+(L)$  and for each  $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$ , we have that*

- (1)  $L - \lambda\Gamma \in \text{Div}_{\mathbb{Q}}(X)$  and it is ample;
- (2)  $(X, \Delta + \lambda\Gamma)$  is an LC pair;
- (3)  $CLC(X, \Delta + \lambda\Gamma) = CLC(X, \Delta)$ .
- (4)  $(X, \Delta + \lambda\Gamma)$  is DLT if  $(X, \Delta)$  is such.

*Moreover  $\Gamma$  can be chosen generically in its linear series.*

*Proof.* By [ELMNP06, Prop. 1.5] there exists  $H$ , an ample  $\mathbb{Q}$ -divisor on  $X$ , and there exists  $m_0 \in \mathbb{N}$  such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H) = Bs(|m_0(L - H)|).$$

Hence, as  $CLC(X, \Delta)$  is a finite set, we can choose a general divisor  $\Gamma$  in  $|m_0(L - H)|$  such that  $\text{Supp}(\Gamma)$  does not contain LC centers of  $(X, \Delta)$ .

Thus there exists  $\lambda_1 > 0$  such that if  $\lambda \in \mathbb{Q} \cap (0, \lambda_1)$ , then  $(X, \Delta + \lambda\Gamma)$  is an LC pair and  $CLC(X, \Delta + \lambda\Gamma) = CLC(X, \Delta)$ .

Moreover it is easy to see that  $(X, \Delta + \lambda\Gamma)$  is DLT if  $(X, \Delta)$  is such.

Now, for all  $\lambda \in \mathbb{Q} \cap (0, 1]$ , we have that

$$L - \lambda\Gamma \sim_{\mathbb{Q}} (1 - \lambda m_0)L + \lambda m_0 H$$

is ample if  $\lambda \leq \frac{1}{m_0}$ . Thus, if we define  $\lambda_0 = \min\{\lambda_1, \frac{1}{m_0}\}$ , we get the thesis.  $\square$

**2.3. Zariski decomposition and birational modifications.** For our purposes we need to extend the classical definition of Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki to some non  $\mathbb{Q}$ -Cartier cases. From now on we will use the following definition:

**Definition 2.4.** Let  $X$  be a normal projective variety and let  $D$  be a Weil  $\mathbb{Q}$ -divisor on  $X$ . We say that  $D$  admits a  $\mathbb{Q}$ -Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki (or a  $\mathbb{Q}$ -CKM Zariski decomposition)  $D = P + N$  if

- $P$  is a  $\mathbb{Q}$ -Cartier divisor and  $N$  is a Weil  $\mathbb{Q}$ -divisor;
- $P$  is nef and  $N$  is effective;
- There exists an integer  $k > 0$  such that  $kP$  is Cartier,  $kD$  is an integral Weil divisor and for every  $m \in \mathbb{N}$  we have an isomorphism

$$H^0(X, \mathcal{O}_X(kmP)) \simeq H^0(X, \mathcal{O}_X(kmD)).$$

Note that  $kmD$  might not be a Cartier divisor but it still makes sense to consider the reflexive sheaf  $\mathcal{O}_X(kmD)$  and its  $H^0$ . In particular if  $D$  is  $\mathbb{Q}$ -Cartier this definition coincides with the one given in the introduction.

**Definition 2.5.** Let  $(X, \Delta)$  be a pair with  $\Delta$  effective. We define the b-divisors  $\mathbf{A}(\Delta)$  and  $\mathbf{L}(\Delta)$ :

For every birational morphism  $f : Z \rightarrow X$ , if  $E$  and  $F$  are effective Weil  $\mathbb{Q}$ -divisors on  $Z$  without common components such that

$$K_Z + E \equiv f^*(K_X + \Delta) + F \quad \text{and} \quad f_*(E - F) = \Delta,$$

we put the trace  $\mathbf{A}(\Delta)_Z = E - F$  and the trace  $\mathbf{L}(\Delta)_Z = E$ .

The following lemma will be very useful to treat the case when a birational pullback of a given divisor admits a Zariski decomposition.

**Lemma 2.6.** *Let  $(X, \Delta)$  be a pair such that  $\Delta$  is effective, let  $D \in \text{Div}_{\mathbb{Q}}(X)$  and let  $a \in \mathbb{Q}$ . If there exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D) = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition, then there exist Weil  $\mathbb{Q}$ -divisors  $D', P', N', \Delta_Z$  such that*

- $\Delta_Z$  is effective;
- $D' = P' + N'$  is a  $\mathbb{Q}$ -CKM Zariski decomposition;
- $\Delta_Z - N' = \mathbf{A}(\Delta)_Z - aN$ , so that in particular  $(Z, \Delta_Z - N')$  is a pair;
- $P' = bP$  for some  $b > 0$ ;
- $t_0 P' - (K_Z + \Delta_Z - N') = P + f^*(aD - (K_X + \Delta))$  for some  $t_0 \in \mathbb{Q}$ .

*In particular if  $D$  is big and  $aD - (K_X + \Delta)$  is nef **or** if  $aD - (K_X + \Delta)$  is big and nef, then*

$$t_0 P' - (K_Z + \Delta_Z - N')$$

*is big and nef.*

*Proof.* Define  $a' = -\min\{0, a\}$  and  $a'' = \max\{0, a\}$ , so that  $a' \geq 0$ ,  $a'' \geq 0$  and  $a = a'' - a'$ . Moreover we can write  $\mathbf{A}(\Delta)_Z = A^+ - A^-$ , where  $A^+$  and  $A^-$  are effective and without common components, so that  $A^-$  is  $f$ -exceptional. We define  $\Delta_Z := A^+ + a'N$ ,  $N' := a''N + A^-$ ,  $P' := (a'' + 1)P$  and  $D' := P' + N'$ .

Then it is immediate that  $\Delta_Z$  is effective,  $\Delta_Z - N' = \mathbf{A}(\Delta)_Z - aN$  and  $P'$  is a positive rational multiple of  $P$ .

Moreover  $P'$  is a nef  $\mathbb{Q}$ -Cartier divisor,  $N'$  is effective and by using the hypothesis and Fujita's lemma we can see that there exists  $k' \in \mathbb{N}$  such that

$$H^0(Z, \mathcal{O}_Z(k'mP')) \simeq H^0(Z, \mathcal{O}_Z(k'mD'))$$

for every  $m \in \mathbb{N}$ , so that  $D' = P' + N'$  is a  $\mathbb{Q}$ -CKM Zariski decomposition.

Now note that

$$\begin{aligned} P + f^*(aD - (K_X + \Delta)) &= (a + 1)P + aN - (K_Z + \mathbf{A}(\Delta)_Z) = \\ &= (a + 1)P + a''N - (K_Z + \mathbf{A}(\Delta)_Z + a'N) = (a + 1)P - (K_Z + \Delta_Z - N') = \\ &= t_0 P' - (K_Z + \Delta_Z - N'), \end{aligned}$$

where  $t_0 = \frac{a+1}{a''+1} \in \mathbb{Q}$ .

□

### 3. $\mathbb{Q}$ -GORENSTEIN CASE

In this section we present some results for  $\mathbb{Q}$ -Gorenstein pairs.

In particular the following proposition treats the case of an LC pair that can be “approximated” with a KLT pair.

**Proposition 3.1.** *Let  $X$  be a normal projective  $\mathbb{Q}$ -Gorenstein variety and let  $\Delta \in \text{Div}_{\mathbb{Q}}(X)$  be effective and such that  $(X, \Delta)$  is an LC pair and  $(X, (1-b)\Delta)$  is a KLT pair for some rational number  $b > 0$ .*

*If  $D \in \text{Div}_{\mathbb{Q}}(X)$  is such that*

- (1)  $D$  is big;
- (2)  $\mathbb{B}_+(D)$  does not contain any LC center of the pair  $(X, \Delta)$ ;
- (3)  $D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition

$$D = P + N;$$

then there exists  $\beta > 0$  such that if

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta,$$

then  $P$  is semiample.

*Proof.* The hypothesis on the pair  $(X, \Delta)$  implies that  $\text{Supp}(\Delta)$  contains all the LC centers of the pair  $(X, \Delta)$ .

Note that  $P$  is big because  $D$  is such. By the definition of  $\mathbb{Q}$ -CKM Zariski decomposition it is easy to see that  $\mathbb{B}_+(P) = \mathbb{B}_+(D)$  and  $\text{Supp}(N) \subseteq \mathbb{B}_+(D)$ . Then, thanks to lemma 2.3, we can find an effective Cartier divisor  $\Gamma$  and a rational number  $\lambda > 0$  such that  $P - \lambda\Gamma$  is ample, the pair  $(X, \Delta + \lambda\Gamma)$  is LC and  $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$ .

Moreover, as  $\text{Supp}(N)$  does not contain any LC center of the pair  $(X, \Delta + \lambda\Gamma)$ , there exists  $\beta \in \mathbb{Q}^+$  such that if  $0 \leq \beta' < \beta$ , then the pair  $(X, \Delta + \lambda\Gamma + \beta'N)$  is LC and  $CLC(X, \Delta + \lambda\Gamma + \beta'N) = CLC(X, \Delta)$ .

This implies that  $\text{Supp}(\Delta)$  contains all the LC centers of the pair  $(X, \Delta + \lambda\Gamma + \beta'N)$ . Hence it is easy to see that for every rational number  $\epsilon \in (0, 1)$ , for every  $\beta' \in [0, \beta]$ , the pair  $(X, (1 - \epsilon)\Delta + \lambda\Gamma + \beta'N)$  is KLT and  $(1 - \epsilon)\Delta + \lambda\Gamma + \beta'N$  is effective.

Now we have that

$$(1 + a)P + aN - (K_X + (1 - \epsilon)\Delta + \lambda\Gamma) = (P - \lambda\Gamma) + (aD - (K_X + \Delta)) + \epsilon\Delta$$

is ample, thanks to the openness of the ample cone, for  $\epsilon > 0$  small enough.

If  $a \geq 0$  we conclude by applying Kawamata's theorem 1.1 to the pair  $(X, (1 - \epsilon)\Delta + \lambda\Gamma)$ .

If  $-\beta < a < 0$  we can apply theorem 1.1 to the pair  $(X, (1 - \epsilon)\Delta + \lambda\Gamma - aN)$ .  $\square$

As an easy corollary of the main result in [Amb05] we state a result in the context of  $P$ -KLT pairs:

**Theorem 3.2.** *Let  $X$  be a normal projective variety and let  $\Delta$  be an effective Weil  $\mathbb{Q}$ -divisor. Let  $D$  be a Weil  $\mathbb{Q}$ -divisor such that*

(1) *There exists a  $\mathbb{Q}$ -CKM Zariski decomposition*

$$D = P + N;$$

(2) *There exist two rational numbers  $a$  and  $t_0$ , with  $a \geq 0$ , such that  $(X, \Delta - aN)$  is a  $P$ -KLT pair and*

$$t_0P - (K_X + \Delta - aN)$$

*is big and nef,*

then  $P$  is semiample.

*Proof.* Let  $B = \Delta - aN$  and let  $B_- = aN$ . Then  $B + B_- = \Delta \geq 0$  and  $t_0P - (K_X + B)$  is big and nef.

Moreover, by definition of  $\mathbb{Q}$ -CKM Zariski decomposition there exists  $k_0 \in \mathbb{N}$  such that  $k_0 > a$ ,  $k_0P$  is a Cartier divisor,  $k_0D$  is integral and

$$H^0(X, \mathcal{O}_X(mk_0P)) \simeq H^0(X, \mathcal{O}_X(mk_0D))$$

for all  $m \in \mathbb{N}$ . But  $\lceil B_- \rceil = \lceil aN \rceil \leq k_0N$ . Hence, for all  $m \in \mathbb{N}$ , we get that

$$H^0(X, \mathcal{O}_X(mk_0P)) \simeq H^0(X, \mathcal{O}_X(mk_0P + \lceil B_- \rceil)).$$

Thus we can apply [Amb05, theorem 2.1] and we get the semiampleness of  $P$ .  $\square$

**Corollary 3.3.** *Let  $(X, \Delta)$  be a pair with  $\Delta$  effective and let  $D \in \text{Div}_{\mathbb{Q}}(X)$ . Consider the following assumptions:*

- (1)  *$D$  is big;*
- (2)  *$aD - (K_X + \Delta)$  is nef for some rational number  $a \in \mathbb{Q}$ ;*
- (2')  *$aD - (K_X + \Delta)$  is big and nef for some rational number  $a \in \mathbb{Q}$ ;*
- (3) *There exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D)$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition*

$$f^*(D) = P + N$$

*and  $(Z, \mathbf{A}(\Delta)_Z - aN)$  is  $P$ -KLT.*

*If  $D$  satisfies 1, 2, and 3, or  $D$  satisfies 2' and 3, then  $P$  is semiample.*

*Proof.* Let us apply lemma 2.6, and consider  $t_0 \in \mathbb{Q}$  and  $D', P', N', \Delta_Z$  Weil  $\mathbb{Q}$ -divisors on  $Z$  as in the lemma. Then  $t_0 P' - (K_Z + \Delta_Z - N')$  is big and nef and  $(Z, \Delta_Z - N')$  is  $P'$ -KLT.

Thus we can apply theorem 3.2 and we are done.  $\square$

If  $X$  is  $\mathbb{Q}$ -Gorenstein, by the previous theorem we get the following:

**Theorem 3.4.** *Let  $(X, \Delta)$  be an LC pair such that  $X$  is  $\mathbb{Q}$ -Gorenstein and  $\Delta$  is effective.*

*Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be such that*

- (1)  *$D$  is big;*
- (2)  *$\mathbb{B}_+(D) \not\supseteq V$ , for every  $V \in \text{CLC}(X, \Delta)$ ;*
- (3)  *$D$  has a  $\mathbb{Q}$ -CKM Zariski decomposition*

$$D = P + N$$

*such that  $\mathbb{B}(P) \cap V = \emptyset$  for every  $V \in \text{CLC}(X, \Delta)$  such that  $V \not\subseteq \text{Supp}(\Delta)$ ;*

*then there exists  $\beta > 0$  such that if*

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta,$$

*then  $P$  is semiample.*

*Proof.* Note that  $P$  is big because  $D$  is such and it is easy to see that  $\mathbb{B}_+(P) = \mathbb{B}_+(D)$  and  $\text{Supp}(N) \subseteq \mathbb{B}_+(D)$ . Then, thanks to lemma 2.3, we can find an effective Cartier divisor  $\Gamma$  and a rational number  $\lambda > 0$  such that  $P - \lambda\Gamma$  is ample, the pair  $(X, \Delta + \lambda\Gamma)$  is LC and  $\text{CLC}(X, \Delta) = \text{CLC}(X, \Delta + \lambda\Gamma)$ .

Now, as  $\text{Supp}(N)$  does not contain any LC center of the pair  $(X, \Delta + \lambda\Gamma)$ , there exists  $\beta \in \mathbb{Q}^+$  such that if  $0 \leq \beta' < \beta$ , then the pair  $(X, \Delta + \lambda\Gamma + \beta'N)$  is LC and  $\text{CLC}(X, \Delta + \lambda\Gamma + \beta'N) = \text{CLC}(X, \Delta)$ .

Suppose  $a > -\beta$  is a rational number such that  $aD - (K_X + \Delta)$  is nef.

Define  $a' = -\min\{0, a\}$ ,  $a'' = \max\{0, a\}$ , so that  $a = a'' - a'$ ,  $a'' \geq 0$ ,  $0 \leq a' < \beta$ . Moreover we define  $\Delta' = \Delta + \lambda\Gamma + a'N$ , so that  $\Delta'$  is effective,  $(X, \Delta')$  is LC and  $\text{CLC}(X, \Delta') = \text{CLC}(X, \Delta)$ . Hence, we get that for every  $\epsilon \in \mathbb{Q}^+$

$$\text{CLC}(X, \Delta' - \epsilon\Delta - a''N) \subseteq \text{CLC}(X, \Delta' - \epsilon\Delta) = \{V \in \text{CLC}(X, \Delta) \text{ such that } V \not\subseteq \text{Supp}(\Delta)\},$$

so that, by hypothesis,  $\mathbb{B}(P)$  does not intersect any LC center of the pair  $(X, \Delta' - \epsilon\Delta - a''N)$ . Moreover

$$\begin{aligned} (1+a)P + a''N - (K_X + \Delta' - \epsilon\Delta) &= (1+a)P + a''N - (K_X + \Delta + \lambda\Gamma + a'N - \epsilon\Delta) = \\ &= (P - \lambda\Gamma) + (aD - (K_X + \Delta)) + \epsilon\Delta \end{aligned}$$



is ample if  $\epsilon$  is sufficiently small thanks to the openness of the ample cone. Thus we obtain the semiampleness of  $P$  by applying theorem 3.2 to the pair  $(X, \Delta' - \epsilon\Delta)$ .  $\square$

#### 4. DLT CASE

The aim of this section is to prove theorem 4.2, by reducing ourselves to the hypotheses of [Fuj07b, theorem 5.1]. In particular the theorem implies Conjecture 1 in the DLT case.

**Definition 4.1.** Let  $(X, \Delta)$  be a pair and let  $L \in \text{Div}_{\mathbb{Q}}(X)$ .

We say that  $L$  is *logbig* for the pair  $(X, \Delta)$  if  $L$  is big and  $L|_V$  is big for every  $V \in \text{CLC}(X, \Delta)$ .

Moreover given an integer  $k \in \{1, \dots, n\}$  we say that  $L$  is *logbig in codimension  $k$*  if  $L$  is big and  $L|_V$  is big for every  $V \in \text{CLC}(X, \Delta)$  such that  $\text{codim}_X V = k$ .

**Theorem 4.2.** Let  $(X, \Delta)$  be an LC pair with  $\Delta$  effective and let  $D \in \text{Div}_{\mathbb{Q}}(X)$ . Consider the following assumptions:

- (1)  $D$  is big;
- (2)  $aD - (K_X + \Delta)$  is nef for some rational number  $a \geq 0$ ;
- (2')  $aD - (K_X + \Delta)$  is big and nef for some rational number  $a \geq 0$ ;
- (3) There exists a projective birational morphism  $f : Z \rightarrow X$  such that

$$f^*(D) = P + N$$

is a  $\mathbb{Q}$ -CKM Zariski decomposition and the pair  $(Z, \mathbf{L}(\Delta)_Z)$  is a DLT pair;

- (4)  $P$  is logbig for the pair  $(Z, \mathbf{L}(\Delta)_Z)$ ;
- (4')  $f^*(aD - (K_X + \Delta))$  is logbig for the pair  $(Z, \mathbf{L}(\Delta)_Z)$ .

If  $D$  satisfies 1,2,3,4 or 2',3,4', then  $P$  is semiample.

*Proof.* Let us apply lemma 2.6 and take  $t_0 \in \mathbb{Q}$  and  $D', P', N', \Delta_Z$  Weil  $\mathbb{Q}$ -divisors on  $Z$  as in the lemma, so that in particular  $D' = P' + N'$  is a  $\mathbb{Q}$ -CKM Zariski decomposition.

Define  $B := \Delta_Z - N' = \mathbf{A}(\Delta)_Z - aN \leq \mathbf{L}(\Delta)_Z$ , so that  $t_0 P' - (K_Z + B) = P + f^*(aD - (K_X + \Delta))$  is big and nef.

Moreover  $t_0 P' - (K_Z + B)$  is logbig for the pair  $(Z, \mathbf{L}(\Delta)_Z)$  because, by hypothesis, its restriction to every LC center of the pair  $(Z, \mathbf{L}(\Delta)_Z)$  is the sum of a big and a nef divisor. Then we have that  $t_0 P' - (K_Z + B)$  is logbig for the pair  $(Z, B)$ , because  $\text{CLC}(Z, B) \subseteq \text{CLC}(Z, \mathbf{L}(\Delta)_Z)$ .

Now thanks to the main theorem of [Sza95], the DLTness of  $(Z, \mathbf{L}(\Delta)_Z)$  implies that every LC center of  $(Z, \mathbf{L}(\Delta)_Z)$  is not contained in  $\text{Sing}(Z) \cup \text{NSNC}(\mathbf{L}(\Delta)_Z)$ .

Thus it is easy to see that every LC center of the pair  $(Z, B)$  is not contained in  $\text{Sing}(Z) \cup \text{NSNC}(B)$ .

Let  $\mu : Z' \rightarrow Z$  be a standard log-resolution of the pair  $(Z, B)$ , so that  $(Z', \mathbf{A}(B)_{Z'})$  is an LC pair,  $Z'$  is smooth and  $\mathbf{A}(B)_{Z'}$  is SNCS.

Now we choose  $k_0 \in \mathbb{N}$  such that  $k_0 P'$  is a Cartier divisor,  $k_0 D'$  is integral and

$$H^0(Z, \mathcal{O}_Z(mk_0 P')) \simeq H^0(Z, \mathcal{O}_Z(mk_0 D'))$$

for all  $m \in \mathbb{N}$ .

Moreover if we write  $\mathbf{A}(B)_{Z'} = (B')_+ - (B')_-$ , where  $(B')_+$  and  $(B')_-$  are effective divisors and they have not common components, then  $\mu_*(\ulcorner (B')_- \urcorner) \leq \ulcorner N' \urcorner$ , because

$\Delta_Z$  is effective. Thus, by the projection formula, we get that for all  $m \in \mathbb{N}$

$$\begin{aligned} h^0(Z', \mathcal{O}_{Z'}(\mu^*(mk_0P') + \ulcorner(B')_{-}\urcorner)) &\leq h^0(Z, \mathcal{O}_Z(mk_0P' + \ulcorner N'\urcorner)) \leq \\ &\leq h^0(Z, \mathcal{O}_Z(mk_0D')) = h^0(Z, \mathcal{O}_Z(mk_0P')) = h^0(Z', \mathcal{O}_{Z'}(\mu^*(mk_0P'))). \end{aligned}$$

Note also that

$$t_0\mu^*(P') - (K_{Z'} + \mathbf{A}(B)_{Z'}) \equiv \mu^*(t_0P' - (K_Z + B))$$

is big and nef, being the birational pullback of a big and nef divisor.

We will prove that  $\mu^*(t_0P' - (K_Z + B))$  is logbig for the pair  $(Z', \mathbf{A}(B)_{Z'})$ : Let  $V \in CLC(Z', \mathbf{A}(B)_{Z'})$ . Then  $\mu(V) \not\subseteq Sing(Z) \cup NSNC(B)$ . Thanks to the choice of  $\mu$  this implies that  $V \not\subseteq exc(\mu)$ , where we denote by  $exc(\mu)$  the exceptional locus of  $\mu$ , that is the complement of the biggest open subset of  $Y$  on which  $\mu$  is an isomorphism. Then  $\mu|_V$  is birational.

Consider the following commutative diagram:

$$\begin{array}{ccc} V & \xhookrightarrow{\quad} & Z' \\ \downarrow \mu|_V & & \downarrow \mu \\ \mu(V) & \xhookrightarrow{\quad} & Z \end{array}$$

We know that  $t_0P' - (K_Z + B)$  is logbig for the pair  $(Z, B)$ , which implies that  $(t_0P' - (K_Z + B))|_{\mu(V)}$  is big.

Then, by birationality of  $\mu|_V$ , we have that  $\mu|_V^*((t_0P' - (K_Z + B))|_{\mu(V)})$  is a big  $\mathbb{Q}$ -divisor on  $V$ .

But, by commutativity of the diagram, we have that

$$\mu|_V^*((t_0P' - (K_Z + B))|_{\mu(V)}) = (\mu^*(t_0P' - (K_Z + B)))|_V.$$

Thus we have proved that  $\mu^*(t_0P' - (K_Z + B))$  is big when restricted to each LC center of the pair  $(Z', \mathbf{A}(B)_{Z'})$ , whence it is logbig for the pair  $(Z', \mathbf{A}(B)_{Z'})$ .

Hence  $t_0\mu^*(P') - (K_{Z'} + \mathbf{A}(B)_{Z'})$  is logbig for the pair  $(Z', \mathbf{A}(B)_{Z'})$ .

Therefore we can apply [Fuj07b, theorem 5.1] to the divisor  $\mu^*(P')$  and the pair  $(Z', \mathbf{A}(B)_{Z'})$ , so that  $\mu^*(P')$  is semiample, which implies that  $P$  is semiample.  $\square$

## 5. LC GENERAL CASE

**Definition 5.1.** Let  $(X, \Delta)$  be a pair and let  $k \in \{1, \dots, n\}$ , then we define

$$\widetilde{Nklt}(X, \Delta) = \bigcup_{\substack{V \in CLC(X, \Delta) \\ V \subseteq NSNC(\Delta) \cup Sing(X)}} V, \quad Nklt_k(X, \Delta) = \bigcup_{\substack{V \in CLC(X, \Delta) \\ \dim V \leq n-k}} V.$$

Note that if  $(X, \Delta)$  is DLT then  $\widetilde{Nklt}(X, \Delta) = \emptyset$  by [Sza95].

**Theorem 5.2.** Let  $(X, \Delta)$  be a pair and suppose that  $\Delta = \sum_{i \in I} d_i D_i$ , where all the  $D_i$ 's are distinct prime divisors and  $d_i \leq 1$  for every  $i \in I$ .

Moreover suppose that  $P \in Div_{\mathbb{Q}}(X)$  and we can write  $\Delta = \Delta_+ - \Delta_-$ , where  $\Delta_+$  and  $\Delta_-$  are effective  $\mathbb{Q}$ -divisors and the following properties are satisfied:

- (1)  $P$  is nef;
- (2)  $t_0P - (K_X + \Delta)$  is ample for some  $t_0 \in \mathbb{Q}^+$ ;

- (3) *There exists  $k_0 \in \mathbb{N}$  such that  $k_0 P$  is a Cartier divisor and for all  $m \in \mathbb{N}$  it holds that*

$$H^0(X, \mathcal{O}_X(mk_0 P)) \simeq H^0(X, \mathcal{O}_X(mk_0 P + \lceil \Delta_- \rceil));$$

- (4)  *$\widetilde{Nklt}(X, \Delta) = \emptyset$ , or  $P|_{\widetilde{Nklt}(X, \Delta)}$  is semiample;*

- (5) *There exists  $\mu : X' \rightarrow X$ , a standard log-resolution of the pair  $(X, \Delta)$  such that  $a(E, X, \Delta) > -2$  for every prime divisor  $E \subseteq X'$ .*

*Then  $P$  is semiample.*

*Proof.* Let  $\mu : X' \rightarrow X$  be as in the hypothesis. Note that  $Nklt(X', \mathbf{A}(\Delta)_{X'}) = \text{Supp}((\mathbf{A}(\Delta)_{X'})^{\geq 1})$ , because  $X'$  is smooth and  $\mathbf{A}(\Delta)_{X'}$  is SNCS.

Now by the ampleness of  $t_0 P - (K_X + \Delta)$ , for all  $\mu$ -exceptional divisors  $E_1, \dots, E_s$  on  $X'$  there exist arbitrarily small coefficients  $\delta_1, \dots, \delta_s \in \mathbb{Q}^+$ , such that

$$\mu^*(t_0 P - (K_X + \Delta)) - \sum_{j=1}^s \delta_j E_j$$

is ample. Then, if  $0 \leq \epsilon \ll 1$  we have that  $\mu^*(t_0 P) - (K_{X'} + (1 - \epsilon)\mathbf{A}(\Delta)_{X'} + \sum_{j=1}^s \delta_j E_j)$  is still ample. For every  $\epsilon$  sufficiently small such that the above condition holds we define

$$\widehat{\Delta}_\epsilon = (1 - \epsilon)\mathbf{A}(\Delta)_{X'} + \sum \delta_j E_j,$$

so that  $\mu^*(mP) - (K_{X'} + \widehat{\Delta}_\epsilon)$  is ample for every integer  $m \geq t_0$  thanks to the nefness of  $P$ . Now we can write

$$\mathbf{A}(\Delta)_{X'} = \sum_{k \in K} c_k X_k + \sum_{l \in L} a_l Y_l - \sum_{m \in M} b_m Z_m,$$

where, for every  $k \in K$ ,  $l \in L$  and  $m \in M$ , we have that  $X_k, Y_l, Z_m$  are pairwise distinct prime divisors, and

$$b_m > 0 \quad \forall m \in M, \quad 0 \leq a_l < 1 \quad \forall l \in L, \quad 1 \leq c_k < 2 \quad \forall k \in K :$$

In fact all the coefficients of  $\mathbf{A}(\Delta)_{X'}$  are smaller than 2 because of the choice of  $\mu$ . Moreover we can suppose that  $\text{exc}(\mu) \subseteq \text{Supp}(\sum X_k + \sum Y_l + \sum Z_m)$ , by considering among the  $Y_l$ 's also the prime  $\mu$ -exceptional divisors not appearing in  $\text{Supp}(\mathbf{A}(\Delta)_{X'})$ , with coefficient 0. Let us define

$$\Delta'_+ := \sum_{k \in K} c_k X_k + \sum_{l \in L} a_l Y_l; \quad \Delta'_- := \sum_{m \in M} b_m Z_m,$$

so that  $\Delta'_+$  and  $\Delta'_-$  are effective, they have no common components and  $\mathbf{A}(\Delta)_{X'} = \Delta'_+ - \Delta'_-$ . Moreover for every  $k \in K$ ,  $l \in L$  and  $m \in M$  we define

$$\gamma_k = \begin{cases} \delta_j & \text{if } X_k = E_j \\ 0 & \text{otherwise} \end{cases} ; \quad \gamma_l = \begin{cases} \delta_j & \text{if } Y_l = E_j \\ 0 & \text{otherwise} \end{cases} ; \quad \gamma_m = \begin{cases} \delta_j & \text{if } Z_m = E_j \\ 0 & \text{otherwise} \end{cases} ,$$

so that we can write

$$\widehat{\Delta}_\epsilon = \sum_{k \in K} ((1 - \epsilon)c_k + \gamma_k) X_k + \sum_{l \in L} ((1 - \epsilon)a_l + \gamma_l) Y_l - \sum_{m \in M} ((1 - \epsilon)b_m - \gamma_m) Z_m.$$

Now we choose  $\epsilon$  and the  $\delta_j$ 's small enough such that the following inequalities hold:

- $c'_k := (1 - \epsilon)c_k + \gamma_k < 2 \quad \forall k \in K$ ;
- $a'_l := (1 - \epsilon)a_l + \gamma_l < 1 \quad \forall l \in L$ ;
- $b'_m := (1 - \epsilon)b_m - \gamma_m > 0 \quad \forall m \in M$ ,

and we define  $\widehat{\Delta} := \widehat{\Delta}_\epsilon$ . Hence  $\widehat{\Delta} = \sum c'_k X_k + \sum a'_l Y_l - \sum b'_m Z_m$ , and

- $0 < c'_k < 2 \quad \forall k \in K$ ;
- $0 \leq a'_l < 1 \quad \forall l \in L$ ;
- $0 < b'_m \leq b_m \quad \forall m \in M$ .

Note that

$$\mu^*(mP) - (K_{X'} + \widehat{\Delta})$$

is ample for every integer  $m \geq t_0$  and  $\widehat{\Delta}$  is a SNCS divisor because  $\text{Supp}(\widehat{\Delta}) \subseteq \text{Supp}(\mathbf{A}(\Delta)_{X'}) \cup \text{exc}(\mu)$ , so that  $Nklt(X', \widehat{\Delta}) = \text{Supp}((\widehat{\Delta})^{\geq 1})$ .

Now we define

$$\widehat{\Delta}_+ := \sum c'_k X_k + \sum a'_l Y_l; \quad \widehat{\Delta}_- := \sum b'_m Z_m,$$

so that  $\widehat{\Delta}_+$  and  $\widehat{\Delta}_-$  are effective and  $\widehat{\Delta} = \widehat{\Delta}_+ - \widehat{\Delta}_-$ .

We claim that  $\mu_* \lceil \widehat{\Delta}_- \rceil \leq \lceil \Delta_- \rceil$ :

In fact  $\widehat{\Delta}_- \leq \Delta'_-$ , so that it suffices to show that  $\mu_* \lceil \Delta'_- \rceil \leq \lceil \Delta_- \rceil$ . In particular we will show that  $\mu_* \Delta'_- \leq \Delta_-$ .

The required inequality holds because, by definition,  $\Delta'_- = \sum_{a(E, X, \Delta) > 0} a(E, X, \Delta) E$ . Hence

$$\mu_*(\Delta'_-) = \sum_{a(\mu_*^{-1} D_i, X, \Delta) > 0} a(\mu_*^{-1} D_i, X, \Delta) D_i = \sum_{d_i < 0} -d_i D_i \leq \Delta_-,$$

because  $\Delta_-$  is effective and  $\Delta_- = \Delta_+ - \Delta$ , so that, for every  $i$ , we have that  $\text{ord}_{D_i} \Delta_- = \text{ord}_{D_i} \Delta_+ - d_i \geq -d_i$ . Thus the claim is proved.

Thanks to the claim, by using the projection formula, we obtain that if  $k_0$  is as in the hypothesis, then

$$h^0(Y, \mathcal{O}_Y(\mu^*(k_0 m P) + \lceil \widehat{\Delta}_- \rceil)) \leq h^0(X, \mathcal{O}_X(k_0 m P + \lceil \Delta_- \rceil))$$

for all  $m \in \mathbb{N}$ . But, by hypothesis,  $h^0(Y, \mathcal{O}_Y(\mu^*(k_0 m P))) = h^0(X, \mathcal{O}_X(k_0 m P)) = h^0(X, \mathcal{O}_X(k_0 m P + \lceil \Delta_- \rceil))$  for all  $m \in \mathbb{N}$ . Therefore, for all  $m \in \mathbb{N}$

$$H^0(Y, \mathcal{O}_Y(\mu^*(k_0 m P))) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(k_0 m P) + \lceil \widehat{\Delta}_- \rceil)).$$

We will show the semiamplessness of  $P$  by applying theorem 2.1 in [Amb05] to the pair  $(X', \widehat{\Delta})$  and the divisor  $\mu^*(P)$ .

In particular, in order to apply the theorem it remains to show that  $\mathbb{B}(\mu^*(P)) \cap Nklt(X', \widehat{\Delta}) = \emptyset$ :

Note that

$$Nklt(X', \widehat{\Delta}) = \text{Supp}((\widehat{\Delta})^{\geq 1}) \subseteq \bigcup_{k \in K} X_k = \text{Supp}((\mathbf{A}(\Delta)_{X'})^{\geq 1}) = Nklt(X', \mathbf{A}(\Delta)_{X'}).$$

Moreover  $Nklt(X', \widehat{\Delta}) \subseteq \text{exc}(\mu)$ :

In fact if  $k \in K$  is such that  $X_k$  is not exceptional, then  $X_k = \mu_*^{-1} G$ , for some prime divisor  $G$  on  $X$ . Then  $c_k = a(\mu_*^{-1} G, X, \Delta) = -\text{ord}_G \Delta \geq -1$ , thanks to the hypotheses on  $\Delta$ . On the other hand  $\gamma_k = 0$  because  $X_k$  is not exceptional, so that  $c'_k = (1 - \epsilon) c_k < c_k \leq 1$ .

Thus we get that  $Nklt(X', \widehat{\Delta}) \subseteq Nklt(X', \mathbf{A}(\Delta)_{X'}) \cap \text{exc}(\mu)$ . Now we define

$$T = \sum_{c'_k \geq 1} X_k,$$

so that  $T$  is reduced and  $T = \text{Supp}((\widehat{\Delta})^{\geq 1}) = \text{Nklt}(X', \widehat{\Delta})$ . In particular  $T \subseteq \text{Nklt}(X', \mathbf{A}(\Delta)_{X'}) \cap \text{exc}(\mu)$ .

Let  $T_0$  be a prime divisor in the support of  $T$ . Then, on the one hand,  $T_0 \subseteq \text{Supp}((\mathbf{A}(\Delta)_{X'})^{\geq 1})$ , that is  $a(T_0, X, \Delta) \leq -1$ , which implies that  $\mu(T_0) \in \text{CLC}(X, \Delta)$ . On the other hand  $T_0 \subseteq \text{exc}(\mu)$  implies that  $\mu(T_0) \subseteq \text{Sing}(X) \cup \text{NSNC}(\Delta)$ , because  $\mu$  is a standard log-resolution of the pair  $(X, \Delta)$ .

Hence we get that  $\mu(T_0) \subseteq \widetilde{\text{Nklt}}(X, \Delta)$ . But the same holds for every component of  $T$ , so that we have

$$\mu(T) \subseteq \widetilde{\text{Nklt}}(X, \Delta).$$

If  $\widetilde{\text{Nklt}}(X, \Delta) = \emptyset$ , then  $\mu(T) = \emptyset$ , so that  $T = \text{Nklt}(X', \widehat{\Delta}) = \emptyset$  and there is nothing to prove. We can thus assume that  $\widetilde{\text{Nklt}}(X, \Delta) \neq \emptyset$ .

Then, as by hypothesis  $P|_{\widetilde{\text{Nklt}}(X, \Delta)}$  is semiample, we get that  $P|_{\mu(T)}$  is semiample.

Now we consider the commutative diagram:

$$\begin{array}{ccc} T & \hookrightarrow & X' \\ \downarrow \mu|_T & & \downarrow \mu \\ \mu(T) & \hookrightarrow & X \end{array}$$

As  $P|_{\mu(T)}$  is semiample, we have that  $\mu^*_{|T}(P|_{\mu(T)})$  is semiample, which implies that  $\mu^*(P)|_T$  is semiample.

Now we claim that  $\ulcorner -\widehat{\Delta} \urcorner = \ulcorner \widehat{\Delta}_- \urcorner - T$ : In fact

$$\ulcorner -\widehat{\Delta} \urcorner = \sum_{m \in M} \ulcorner b'_m \urcorner Z_m + \sum_{k \in K} \ulcorner -c'_k \urcorner X_k + \sum_{l \in L} \ulcorner -a'_l \urcorner Y_l.$$

But, for all  $l \in L$ , we have that  $0 \geq -a'_l > -1$ , so that  $\ulcorner -a'_l \urcorner = 0$ . Moreover for all  $k \in K$ ,  $0 > -c'_k > -2$ , so that

$$\ulcorner -c'_k \urcorner = \begin{cases} -1 & \text{if } c'_k \geq 1 \\ 0 & \text{if } c'_k < 1 \end{cases}$$

Thus

$$\ulcorner -\widehat{\Delta} \urcorner = \sum_{m \in M} \ulcorner b'_m \urcorner Z_m - \sum_{c'_k \geq 1} X_k = \ulcorner \widehat{\Delta}_- \urcorner - T.$$

Take  $k_1 \in \mathbb{N}$  such that  $k_1 > t_0$  and  $k_1$  is a multiple of  $k_0$ , so that  $k_1 P$  is a Cartier divisor and

$$H^0(Y, \mathcal{O}_Y(\mu^*(k_1 m P))) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(k_1 m P) + \ulcorner \widehat{\Delta}_- \urcorner))$$

for every  $m \in \mathbb{N}$ . Let us consider, for every  $k \in k_1 \mathbb{N}$ , the following commutative diagram:

$$\begin{array}{ccc} H^0(X', \mathcal{O}_{X'}(\mu^*(k P) + \ulcorner \widehat{\Delta}_- \urcorner)) & \xrightarrow{\beta_k} & H^0(T, \mathcal{O}_T(\mu^*(k P)|_T + \ulcorner \widehat{\Delta}_- \urcorner|_T)) \\ \uparrow \simeq & & \uparrow i_k \\ H^0(X', \mathcal{O}_{X'}(\mu^*(k P))) & \xrightarrow{\alpha_k} & H^0(T, \mathcal{O}_T(\mu^*(k P)|_T)) \end{array}$$

where the vertical arrow on the left is an isomorphism thanks to the choice of  $k_1$ .

Note that  $i_k$  is injective for every  $k \in k_1 \mathbb{N}$  because  $\ulcorner \widehat{\Delta}_- \urcorner|_T$  is effective:

In fact  $\lceil \widehat{\Delta}_- \rceil$  is effective and  $\text{Supp}(\lceil \widehat{\Delta}_- \rceil) = \text{Supp}(\widehat{\Delta}_-) = \cup Z_m$  does not contain any component of  $T$ .

Let us prove that  $\beta_k$  is surjective for every  $k \in k_1\mathbb{N}$ . In particular we prove that  $H^1(X', \mathcal{O}_{X'}(\mu^*(kP) + \lceil \widehat{\Delta}_- \rceil - T)) = 0$ :

Note that  $\mu^*(kP) - (K_{X'} + \widehat{\Delta})$  is ample, thanks to the choice of  $k_1$ , and  $\{\mu^*(kP) - (K_{X'} + \widehat{\Delta})\} = \{-\widehat{\Delta}\}$  is SNCS. Then, by Kawamata-Viehweg vanishing theorem (see [Laz04, 9.1.20]), we get that  $H^1(X', \mathcal{O}_{X'}(\mu^*(kP) + \lceil -\widehat{\Delta} \rceil)) = 0$ .

But  $\lceil -\widehat{\Delta} \rceil = \lceil \widehat{\Delta}_- \rceil - T$ . Then  $H^1(X', \mathcal{O}_{X'}(\mu^*(kP) + \lceil \widehat{\Delta}_- \rceil - T)) = 0$ , as required.

By the commutativity of the diagram, the surjectivity of  $\beta_k$  implies that  $i_k$  is surjective, that is  $i_k$  is an isomorphism. Thus  $\alpha_k$  is also surjective for every  $k \in k_1\mathbb{N}$ . But  $\mu^*(P)|_T$  is semiample, whence there exists  $k_2 \in k_1\mathbb{N}$  such that  $\mu^*(k_2P)|_T$  is base point free.

Then the surjectivity of  $\alpha_{k_2}$  implies that  $Bs(\mu^*(k_2P)) \cap T = \emptyset$ . Therefore  $\mathbb{B}(\mu^*(P)) \cap \text{Nklt}(X', \widehat{\Delta}) = \emptyset$ .  $\square$

**Corollary 5.3.** *Let  $(X, \Delta)$  be a pair such that  $\Delta$  is effective.*

*Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be such that*

- (1)  *$D$  is big;*
- (2)  *$aD - (K_X + \Delta)$  is nef for some  $a \in \mathbb{Q}$ ;*
- (3) *There exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D) = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition and*
  - *$(Z, \mathbf{A}(\Delta)_Z - aN)$  is an LC pair;*
  - *$\mathbb{B}_+(f^*(D))$  does not contain any LC center of the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ ;*
  - *$\widehat{\text{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$ , or  $P|_{\widehat{\text{Nklt}}(Z, \mathbf{A}(\Delta)_Z - aN)}$  is semiample.*

*Then  $P$  is semiample.*

We remark that if  $a \geq 0$  the LCness of the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$  holds if we suppose that  $(X, \Delta)$  is an LC pair.

*Proof.* Let us apply lemma 2.6 and consider  $t_0, D', P', N', \Delta_Z$  as in the lemma, so that  $t_0P' - (K_Z + \Delta_Z - N')$  is big and nef.

Note that  $\mathbb{B}_+(P') = \mathbb{B}_+(P) = \mathbb{B}_+(f^*(D))$ . Hence we can apply lemma 2.3 to the big and nef  $\mathbb{Q}$ -divisor  $P'$  and to the pair  $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$  and we find a Cartier divisor  $\Gamma$  and a rational number  $\lambda > 0$  such that  $P' - \lambda\Gamma$  is ample,  $(Z, \Delta_Z - N' + \lambda\Gamma)$  is LC and  $CLC(Z, \Delta_Z - N' + \lambda\Gamma) = CLC(Z, \Delta_Z - N')$ .

Furthermore, we can choose  $\Gamma$  generically in its linear series and we have that  $Bs(|\Gamma|) = \mathbb{B}_+(P')$ . Then, by Bertini's theorem, we can suppose that, outside  $\mathbb{B}_+(P')$ ,  $\Gamma$  is smooth and it intersects  $\Delta_Z - N'$  in a simple normal crossing way.

Let us put  $B = \Delta_Z - N' + \lambda\Gamma$ . We will show that the pair  $(Z, B)$  and the  $\mathbb{Q}$ -Cartier divisor  $P'$  satisfy the hypotheses of theorem 5.2.

First of all we have that  $(t_0 + 1)P' - (K_Z + B) = (P' - \lambda\Gamma) + (t_0P' - (K_Z + \Delta_Z - N'))$  is ample, so that property 2 holds.

By the LCness of the pair  $(Z, B)$  we get that all the coefficients of  $B$  are less than or equal to 1 and property 5 holds. Moreover property 1 is trivially verified and property 3 follows by the definition of  $\mathbb{Q}$ -CKM Zariski decomposition because  $\Delta_Z$  is effective.

In order to prove that property 4 holds we will show that  $\widetilde{Nklt}(Z, B) \subseteq \widetilde{Nklt}(Z, \Delta_Z - N') = \widetilde{Nklt}(Z, \mathbf{A}(\Delta)_Z - aN)$ , so that we can use the hypothesis of the corollary: By the choice of  $\Gamma$  we have that  $CLC(Z, \Delta_Z - N') = CLC(Z, B)$  and  $NSNC(B) \subseteq NSNC(\Delta_Z - N') \cup \mathbb{B}_+(P')$ .

Then, if  $V \in CLC(Z, B)$  and  $V \subseteq Sing(Z) \cup NSNC(B)$ , we get that  $V \in CLC(Z, \Delta_Z - N')$  and  $V \subseteq Sing(Z) \cup NSNC(\Delta_Z - N') \cup \mathbb{B}_+(P')$ . This implies that  $V \subseteq Sing(Z) \cup NSNC(\Delta_Z - N')$ . Hence  $V \subseteq \widetilde{Nklt}(Z, \Delta_Z - N')$ , and we get the required inclusion. Therefore we can apply theorem 5.2.  $\square$

**Theorem 5.4.** *Let  $(X, \Delta)$  be an LC pair, with  $\dim X \geq 2$ . Suppose that  $P \in Div_{\mathbb{Q}}(X)$  and we can write  $\Delta = \Delta_+ - \Delta_-$ , where  $\Delta_+$  and  $\Delta_-$  are effective  $\mathbb{Q}$ -divisors, and the following are satisfied:*

- (1)  $P$  is nef;
- (2)  $t_0P - (K_X + \Delta)$  is nef for some  $t_0 \in \mathbb{Q}^+$ ;
- (3) There exists  $k_0 \in \mathbb{N}$  such that  $k_0P$  is a Cartier divisor and for all  $m \in \mathbb{N}$  we have

$$H^0(X, \mathcal{O}_X(mk_0P)) \simeq H^0(X, \mathcal{O}_X(mk_0P + \lceil \Delta_- \rceil));$$

- (4)  $Nklt_2(X, \Delta) = \emptyset$ , or  $P|_{Nklt_2(X, \Delta)}$  is semiample.
- (5)  $P$  is logbig in codimension 1 for the pair  $(X, \Delta)$ , or  $t_0P - (K_X + \Delta)$  is logbig in codimension 1 for the pair  $(X, \Delta)$

Then  $P$  is semiample.

*Proof.* Let

$$L = \begin{cases} P & \text{if } P \text{ is logbig in codimension 1 for the pair } (X, \Delta) \\ t_0P - (K_X + \Delta) & \text{otherwise} \end{cases}$$

Then  $L$  is nef and logbig in codimension 1 for the pair  $(X, \Delta)$ , so that  $\mathbb{B}_+(L)$  does not contain any divisorial LC center of the pair  $(X, \Delta)$ , because given a prime divisor  $E$  on  $X$ ,  $P|_E$  is big if and only if  $\mathbb{B}_+(P) \not\supseteq E$  (see [Laz04, 10.3.6]).

By [ELMNP06, Prop. 1.5] there exists  $H \in Div_{\mathbb{Q}}(X)$  ample and there exists  $m_0 \in \mathbb{N}$  such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H) = Bs(|m_0(L - H)|).$$

Hence, we can choose a general divisor  $\Gamma$  in  $|m_0(L - H)|$  such that  $\text{Supp}(\Gamma)$  does not contain any divisorial LC center of  $(X, \Delta)$ . Note that we have

$$L - \lambda\Gamma \sim_{\mathbb{Q}} (1 - \lambda m_0)L + \lambda m_0 H$$

is ample if  $\lambda \in (0, \frac{1}{m_0}]$ . because  $L$  is nef and  $H$  is ample.

Now, for every  $\lambda \in (0, \frac{1}{m_0}]$ , let us define  $\Delta_\lambda = \Delta + \lambda\Gamma$ . We will prove that there exists  $\lambda_0 \in \mathbb{Q}^+$  such that if  $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$ , then  $P$  and the pair  $(X, \Delta_\lambda)$  satisfy the hypotheses of the theorem 5.2. First of all note that

$$\begin{aligned} (t_0 + 1)P - (K_X + \Delta_\lambda) &= (t_0 + 1)P - (K_X + \Delta + \lambda\Gamma) = P + (t_0P - (K_X + \Delta)) - \lambda\Gamma = \\ &= \begin{cases} L - \lambda\Gamma + (t_0P - (K_X + \Delta)) & \text{if } P \text{ is logbig in codimension 1} \\ P + (L - \lambda\Gamma) & \text{otherwise} \end{cases} \end{aligned}$$

is ample in both cases for every  $\lambda \in (0, \frac{1}{m_0}]$ . Now let us define

$$(\Delta_\lambda)_+ := \Delta_+ + \lambda\Gamma; \quad (\Delta_\lambda)_- := \Delta_-.$$

Then  $\Delta_\lambda = (\Delta_\lambda)_+ - (\Delta_\lambda)_-$ , and  $(\Delta_\lambda)_+$  and  $(\Delta_\lambda)_-$  are effective  $\mathbb{Q}$ -divisors for every  $\lambda > 0$ , because  $\Gamma$  is effective. Moreover note that, with these definitions, hypotheses 1 and 3 of theorem 5.2 are trivially verified.

Now take a rational number  $\lambda' > 0$  such that  $\text{Supp}(\Delta) + \text{Supp}(\Gamma) = \text{Supp}(\Delta + \lambda'\Gamma)$  for every  $\lambda \in (0, \lambda')$ . and let  $\mu : X' \rightarrow X$  be a standard log-resolution of the pair  $(X, \Delta + \lambda'\Gamma)$ . For every prime divisor  $E \subseteq X'$  we have that

$$a(E, X, \Delta_\lambda) = a(E, X, \Delta + \lambda\Gamma) = a(E, X, \Delta) - \lambda \text{ord}_E(\mu^*(\Gamma)),$$

where  $a(E, X, \Delta) \geq -1$  because  $(X, \Delta)$  is an LC pair.

Suppose  $E$  is a divisor on  $X'$  such that  $E$  is not  $\mu$ -exceptional and  $a(E, X, \Delta) = -1$ . Then  $\mu(E)$  is a divisorial LC center of  $(X, \Delta)$ , so that  $\text{ord}_{\mu(E)}\Gamma = 0$ , that is  $\text{ord}_E(\mu^*(\Gamma)) = 0$ , which implies  $a(E, X, \Delta_\lambda) = -1$ .

Now define

$$\lambda_1 := \min_{\substack{\text{ord}_E(\mu^*(\Gamma)) > 0 \\ a(E, X, \Delta) > -1}} \left\{ \frac{1 + a(E, X, \Delta)}{\text{ord}_E(\mu^*(\Gamma))}, 1 \right\}.$$

Then  $\lambda_1 > 0$  and, if  $\lambda \in \mathbb{Q} \cap (0, \lambda_1)$ , we have that  $a(E, X, \Delta_\lambda) > -1$  for every prime divisor  $E \subseteq X'$  such that  $a(E, X, \Delta) > -1$ .

Define

$$\lambda_2 := \min_{\substack{\text{ord}_E(\mu^*(\Gamma)) > 0 \\ a(E, X, \Delta) = -1}} \left\{ \frac{2 + a(E, X, \Delta)}{\text{ord}_E(\mu^*(\Gamma))}, 1 \right\}.$$

Then  $\lambda_2 > 0$  and, if  $\lambda \in \mathbb{Q} \cap (0, \lambda_2)$ , we have that  $a(E, X, \Delta_\lambda) > -2$  for every prime divisor  $E \subseteq X'$  such that  $a(E, X, \Delta) = -1$ .

We put  $\lambda_0 = \min\{\lambda', \lambda_1, \lambda_2, \frac{1}{m_0}\}$ , so that if  $\lambda \in \mathbb{Q} \cap (0, \lambda_0)$  then  $(X, \Delta_\lambda)$  satisfies hypothesis 5 of theorem 5.2.

Furthermore we can write

$$\Delta_\lambda = \sum -a(\mu_*^{-1}B_i, X, \Delta_\lambda)B_i,$$

where the  $B_i$ 's are distinct prime divisors on  $X$ . By definition, for every  $i$ ,  $\mu_*^{-1}B_i$  is not an exceptional divisor, so that, it follows by the previous calculation that  $-a(\mu_*^{-1}B_i, X, \Delta_\lambda) \leq 1$ .

Now let us consider

$$\mathbf{A}(\Delta)_{X'} = \sum_{E \subseteq X'} -a(E, X, \Delta)E; \quad \mathbf{A}(\Delta_\lambda)_{X'} = \sum_{E \subseteq X'} -a(E, X, \Delta_\lambda)E,$$

Thanks to the choice of  $\mu$  and  $\lambda$  we have that they are both SNCS. Let us put

$$F := \sum_{a(E, X, \Delta_\lambda) < -1} (-a(E, X, \Delta_\lambda) - 1)E;$$

$$\tilde{\Delta} := \mathbf{A}(\Delta_\lambda)_{X'} - F = \sum_{a(E, X, \Delta_\lambda) \geq -1} -a(E, X, \Delta_\lambda)E + \sum_{a(E, X, \Delta_\lambda) < -1} E.$$

Then we have that  $F$  is effective,  $\text{Supp}(\tilde{\Delta}) \subseteq \text{Supp}(\mathbf{A}(\Delta_\lambda)_{X'})$  and all the coefficients of  $\tilde{\Delta}$  are less than or equal to 1. In particular the pair  $(X', \tilde{\Delta})$  is LC.

Moreover, by the previous calculations, we have that  $F$  is exceptional,  $\text{Supp}(F) \subseteq \text{Supp}((\mathbf{A}(\Delta)_{X'})^{-1})$  and  $\tilde{\Delta}^{-1} = (\mathbf{A}(\Delta)_{X'})^{-1}$ .



Let us show that  $Nklt_2(X, \Delta_\lambda) \subseteq Nklt_2(X, \Delta)$ :

Let  $V$  be an LC center of the pair  $(X, \Delta_\lambda)$  of codimension greater than one. Then  $V = \mu(W)$  for some  $W \in CLC(X, \mathbf{A}(\Delta_\lambda)_{X'})$ .

If  $W \not\subseteq \text{Supp}(F)$ , then  $W \in CLC(X', \tilde{\Delta})$ , whence  $W$  is an irreducible component of a finite intersection of prime divisors in the support of  $\tilde{\Delta}^{=1} = (\mathbf{A}(\Delta)_{X'})^{=1}$ .

Hence  $W \in CLC(X', \mathbf{A}(\Delta)_{X'})$ , which implies that  $V = \mu(W) \in CLC(X, \Delta)$ , so that  $V \subseteq Nklt_2(X, \Delta)$ , because the codimension of  $V$  is greater than 1.

If  $W \subseteq \text{Supp}(F)$  then there exists a prime divisor  $F_0 \subseteq \text{Supp}(F)$  such that  $W \subseteq F_0$ . Then  $F_0 \subseteq \text{Supp}(F) \subseteq \text{Supp}((\mathbf{A}(\Delta)_{X'})^{=1})$ . Hence  $F_0 \in CLC(X', \mathbf{A}(\Delta)_{X'})$ , so that  $\mu(F_0) \in CLC(X, \Delta)$ . Moreover  $\text{codim} \mu(F_0) \geq 2$ , because  $F_0$  is exceptional. Thus

$$V = \mu(W) \subseteq \mu(F_0) \subseteq Nklt_2(X, \Delta).$$

This shows that  $\widetilde{Nklt}(X, \Delta_\lambda) \subseteq \widetilde{Nklt}_2(X, \Delta_\lambda) \subseteq Nklt_2(X, \Delta)$ , which implies, by the hypotheses, that  $\widetilde{Nklt}(X, \Delta_\lambda) = \emptyset$  or  $P|_{\widetilde{Nklt}(X, \Delta_\lambda)}$  is semiample.

Therefore all the hypotheses of theorem 5.2 are satisfied and we get the semiamplicity of  $P$ .  $\square$

**Corollary 5.5.** *Let  $(X, \Delta)$  be a pair with  $\Delta$  effective and  $\dim X \geq 2$  and let  $a \in \mathbb{Q}$ . Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be such that:*

- (1)  $D$  is big
- (2)  $aD - (K_X + \Delta)$  is nef;
- (3) *There exists a projective birational morphism  $f : Z \rightarrow X$  such that*

$$f^*(D) = P + N$$

*is a  $\mathbb{Q}$ -CKM Zariski decomposition and*

- $(Z, \mathbf{A}(\Delta)_Z - aN)$  is an LC pair;
- $\mathbb{B}_+(f^*(D))$  does not contain divisorial LC centers of the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ ;
- $Nklt_2(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$ , or  $P|_{Nklt_2(Z, \mathbf{A}(\Delta)_Z - aN)}$  is semiample;

*Then  $P$  is semiample.*

Note that in the case  $a \geq 0$  we can just assume that the pair  $(X, \Delta)$  is LC in order to have the LCness of the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ .

*Proof.* Thanks to lemma 2.6 the corollary follows by applying theorem 5.4 to the  $\mathbb{Q}$ -Cartier divisor  $P'$  and to the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ .  $\square$

**Corollary 5.6.** *Let  $(X, \Delta)$  be a pair such that  $\Delta$  is effective and  $\dim X \leq 3$ , let  $a \in \mathbb{Q}$ . Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be such that*

- (1)  $D$  is big;
- (2)  $aD - (K_X + \Delta)$  is nef;
- (3) *There exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D) = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition and*
  - $(X, \Delta)$  is an LC pair and  $a \geq 0$  (resp.  $(Z, \mathbf{A}(\Delta)_Z - aN)$  is an LC pair);
  - $P$  is logbig for the pair  $(Z, \mathbf{A}(\Delta)_Z)$  (resp.  $P$  is logbig for the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ ).

*Then  $P$  is semiample.*

*Proof.* Begin by noting that if  $\dim X \leq 1$  then the theorem is trivial because every big divisor on a curve is ample. We can thus assume that  $2 \leq \dim X \leq 3$ .

Note also that if  $a \geq 0$  and  $(X, \Delta)$  is LC then  $(Z, \mathbf{A}(\Delta)_Z - aN)$  is LC and  $CLC(Z, \mathbf{A}(\Delta)_Z - aN) \subseteq CLC(Z, \mathbf{A}(\Delta)_Z)$ . Thus we can assume that  $P$  is logbig for the LC pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ .

Hence we get that  $\mathbb{B}_+(P)$  does not contain divisorial LC centers of the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$  (see [Laz04, 10.3.6]), so that the same holds for  $\mathbb{B}_+(f^*(D))$ .

Then, in order to apply corollary 5.5, it just remains to show that  $P|_{Nklt_2(Z, \mathbf{A}(\Delta)_Z - aN)}$  is semiample if  $Nklt_2(Z, \mathbf{A}(\Delta)_Z - aN) \neq \emptyset$ .

Let  $C$  be a connected component of  $Nklt_2(Z, \mathbf{A}(\Delta)_Z - aN)$ . Then, by hypothesis, we have that  $0 \leq \dim C \leq 1$ .

If  $\dim C = 0$  then  $P|_C$  is trivially semiample. If  $\dim C = 1$  then we can write  $C = \cup_{j=1}^k C_j$ , where the  $C_j$ 's are irreducible curves.

Then we have that  $C_j \in CLC(Z, \mathbf{A}(\Delta)_Z - aN)$  for every  $j \in \{1, \dots, k\}$ , so that  $P|_{C_j}$  is big, because  $P$  is logbig for the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ .

But, as  $C_j$  is an irreducible curve, this implies that  $P|_{C_j}$  is ample for every  $j = \{1, \dots, k\}$ . Hence  $P|_C$  is ample, so that in particular it is semiample.  $\square$

**5.1. Dimension 4.** In this subsection we show in theorem 5.7 that Conjecture 2 holds if we assume some strong standard conjectures in the field of the Minimal Model Program. By using that these conjectures hold true in low dimension we obtain Conjecture 2 in dimension less than or equal to 4 (cf. corollary 5.8).

Before stating the theorems let us fix some notation and definitions:

- We say that a pair  $(X, \Delta)$  is of *log-general type* if  $K_X + \Delta \in \text{Div}_{\mathbb{Q}}(X)$  is big;
- We refer to [KM00, definition 3.50] for the definition of *minimal model* of a DLT pair.
- We refer to [Fuj00, Definition 1.1] for the definition of *semi log canonical* (or *sLC*)  $n$ -fold:
- We say that *sLC-abundance* holds in dimension  $n$  if for every sLC  $n$ -fold  $(X, \Delta)$  such that  $K_X + \Delta$  is nef we have that  $K_X + \Delta$  is semiample.

**Theorem 5.7.** *Let  $(X, \Delta)$  be an effective LC pair of dimension  $n$ . Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be such that*

- (1)  $D$  is big;
- (2)  $aD - (K_X + \Delta)$  is nef for some rational number  $a \geq 0$ ;
- (3)  $D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $D = P + N$ ;
- (4)  $\mathbb{B}_+(D)$  does not contain any LC center of the pair  $(X, \Delta)$ ;

*Also suppose that minimal models exist for every  $\mathbb{Q}$ -factorial DLT pair of dimension  $n$  of log-general type and that sLC-abundance holds in dimension  $n - 1$ .*

*Then  $P$  is semiample.*

*Proof.* Note that  $\mathbb{B}_+(P) = \mathbb{B}_+(D)$ . Then we can apply lemma 2.3 to  $P$  and we find a Cartier divisor  $\Gamma$  and a rational number  $\lambda > 0$  such that  $P - \lambda\Gamma$  is ample,  $(X, \Delta + \lambda\Gamma)$  is LC and  $CLC(X, \Delta) = CLC(X, \Delta + \lambda\Gamma)$ .

Then  $aD + P - (K_X + \Delta + \lambda\Gamma) = (aD - (K_X + \Delta)) + (P - \lambda\Gamma)$  is ample.

Thus if  $D' = aD + P$ , then  $D'$  is big and  $D'$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $D' = P' + N'$ , where  $P' := (a+1)P$  and  $N' := aN$ . Furthermore  $D' - (K_X + \Delta + \lambda\Gamma)$  is ample,  $P$  is semiample if and only if  $P'$  is such and  $\mathbb{B}_+(D) = \mathbb{B}_+(D')$ .

This implies that, if we replace  $D$  by  $D'$  and  $\Delta$  by  $\Delta + \lambda\Gamma$ , we can suppose that  $D - (K_X + \Delta)$  is ample.

Hence by [KM00, lemma 5.17] there exists an effective ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $H$  such that

$$D - (K_X + \Delta) \sim_{\mathbb{Q}} H,$$

and  $(X, \Delta + H)$  is an LC pair.

In other words, if we put  $\Delta_0 := \Delta + H$ , then  $D \sim_{\mathbb{Q}} K_X + \Delta_0$  and  $(X, \Delta_0)$  is an LC pair.

Therefore, we are reduced to show that if  $K_X + \Delta = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition,  $K_X + \Delta$  is big and  $(X, \Delta)$  is LC, then  $P$  is semiample. Moreover up to performing a DLT blow-up (see [Fuj09b, theorem 10.4]) we can suppose that  $X$  is  $\mathbb{Q}$ -factorial and the pair  $(X, \Delta)$  is DLT.

Then by hypothesis we have a minimal model  $(X', \Delta')$  of the pair  $(X, \Delta)$ , so that there exists  $\phi : X \dashrightarrow X'$  a birational map,  $\Delta' = \phi_*(\Delta)$ ,  $K_{X'} + \Delta'$  is nef and  $(X', \Delta')$  is LC. If we resolve the indeterminacies of  $\phi$  we find two birational morphisms  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  such that

$$f^*(K_X + \Delta) = g^*(K_{X'} + \Delta') + E,$$

where  $E$  is  $g$ -exceptional and effective.

Hence, by Fujita's lemma this is a  $\mathbb{Q}$ -CKM Zariski decomposition of  $f^*(K_X + \Delta)$ . By the uniqueness of the  $\mathbb{Q}$ -CKM Zariski decomposition for big divisors this implies that  $f^*(P) = g^*(K_{X'} + \Delta')$ . Thus we are reduced to prove that  $K_{X'} + \Delta'$  is semiample. Note that we can assume that the pair  $(X', \Delta')$  is DLT by performing again a DLT blow-up if necessary.

Let  $V = \text{Nklt}(X', \Delta')$ . Then there exists a  $\mathbb{Q}$ -divisor  $\Delta'_V$  on  $V$  such that

$$(K_{X'} + \Delta')|_V = K_V + \Delta'_V$$

and  $(V, \Delta'_V)$  is an sLC  $(n-1)$ -fold (see for example [Fuj00, Remark 1.2(3)]).

Hence by sLC-abundance we have that  $(K_{X'} + \Delta')|_V = K_V + \Delta'_V$  is semiample. Moreover for every sufficiently divisible  $m \geq 2$  we have that

$$H^1(X', \mathcal{I}_V(m(K_{X'} + \Delta'))) = 0$$

by Nadel vanishing (see [Laz04, theorem 9.4.17]), because  $K_{X'} + \Delta'$  is big and nef. Therefore we can lift sections and we find that  $\mathbb{B}(K_{X'} + \Delta') \cap \text{Nklt}(X', \Delta') = \emptyset$ .

Thus  $K_{X'} + \Delta'$  is semiample (see for example theorem 3.2) and we are done.  $\square$

**Corollary 5.8.** *Let  $(X, \Delta)$  be an LC pair of dimension less than or equal to 4. If  $D \in \text{Div}_{\mathbb{Q}}(X)$  is such that*

- (1)  $D$  is big;
- (2)  $aD - (K_X + \Delta)$  is nef for some rational number  $a \geq 0$ ;
- (3)  $D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $D = P + N$ ;
- (4)  $\mathbb{B}_+(D)$  does not contain any LC center of the pair  $(X, \Delta)$ ;

*Then  $P$  is semiample.*

*Proof.* sLC-abundance in dimension 3 holds by [Fuj00, theorem 0.1], while every DLT  $\mathbb{Q}$ -factorial pair of dimension 4 of log-general type has a minimal model by [AHK07, corollary 3.6]. Hence we can apply theorem 5.7 and we are done.  $\square$

## 6. RELATIVELY DLT CASE

### 6.1. Relatively DLT pairs.

**Definition 6.1.** Let  $(X, \Delta)$  be a pair, with  $\Delta = \sum a_i D_i$ , where the  $D_i$ 's are distinct prime divisors and  $a_i \in \mathbb{Q}$  for every  $i$ . Suppose  $S \subseteq X$  is a closed subset.

We say that  $(X, \Delta)$  is a *S-DLT pair* if

- (1)  $V \not\subseteq \text{Sing}(X) \cup \text{NSNC}(\Delta)$  for every  $V \in \text{CLC}(X, \Delta, S)$ ;
- (2)  $a_i \leq 1$  for every  $i$  such that  $D_i \cap S \neq \emptyset$ ;

Now let  $D \in \text{Div}_{\mathbb{Q}}(X)$ . We say that  $(X, \Delta)$  is a *D-DLT pair* if  $(X, \Delta)$  is a  $\mathbb{B}(D)$ -DLT pair.

**Remark 6.2.** Let  $S \subseteq X$  be a closed subset. Then it is immediate to see that a S-KLT pair is S-DLT. Moreover by [Sza95] a DLT pair is S-DLT.

**Lemma 6.3.** Let  $(X, \Delta)$  be a pair and let  $D \in \text{Div}_{\mathbb{Q}}(X)$ . Suppose  $m \in \mathbb{N}$  is such that  $mD$  is a Cartier divisor and, set-theoretically,  $\text{Bs}(|mD|) = \mathbb{B}(D)$ .

If  $(X, \Delta)$  is a D-DLT pair and  $\mathbb{B}(D)$  does not contain any LC center of the pair  $(X, \Delta)$ , then there exists a common log-resolution of  $(X, \Delta)$  and of the linear series  $|mD|$ , say  $\mu : Y \rightarrow X$ , such that

- (1)  $a(E, X, \Delta) > -1$  for every  $\mu$ -exceptional prime divisor  $E \subseteq Y$  such that  $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$ ;
- (2)  $a(E, X, \Delta) \geq -1$  for every non- $\mu$ -exceptional prime divisor  $E \subseteq Y$  such that  $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$ ;
- (3)  $\mu$  is a composition of blowings-up of smooth subvarieties of codimension greater than 1

*Proof.* Fix  $m \in \mathbb{N}$  as in the hypothesis and consider  $\mu : Y \rightarrow X$  a log-resolution of  $(X, \Delta)$  and  $|mD|$  such that

- $\mu$  is an isomorphism on  $Y \setminus \mu^{-1}(\text{Bs}(|mD|) \cup \text{Sing}(X) \cup \text{NSNC}(\Delta))$ ;
- $\mu$  is a composition of blowings-up of smooth subvarieties of codimension greater than 1, up to isomorphisms.

The existence of such a resolution follows from the existence of standard log-resolutions of pairs and ideals (see remark 2.2).

It is easy to see that  $\mu$  satisfies all the given conditions.  $\square$

In the following lemmas we prove some good properties of S-DLT pairs:

**Lemma 6.4.** Let  $(X, \Delta)$  be a pair and let  $S \subseteq X$  be a closed subset.

If  $(X, \Delta)$  is a S-DLT pair, then  $\text{CLC}(X, \Delta, S)$  is a finite set.

*Proof.* Write  $\Delta = \sum a_i D_i$ , where the  $a_i$ 's are rational numbers and the  $D_i$ 's are distinct prime divisors on  $X$ . Let  $\mu : Y \rightarrow X$  be a standard log-resolution of the pair  $(X, \Delta)$ .

If  $E \subseteq Y$  is a  $\mu$ -exceptional prime divisor, then  $\mu(E) \subseteq \text{Sing}(X) \cup \text{NSNC}(\Delta)$ , so that, by hypothesis  $\mu(E) \notin \text{CLC}(X, \Delta, S)$ . Thus, if  $\mu(E) \cap S \neq \emptyset$ , then  $a(E, X, \Delta) > -1$ . Now define

$$\Delta_Y := - \sum a(E, X, \Delta) E = - \sum a(\mu_*^{-1}(D_i), X, \Delta) \mu_*^{-1}(D_i) - \sum_{E \text{ exc.}} a(E, X, \Delta) E,$$

so that for every  $V \in \text{CLC}(X, \Delta, S)$  we have that  $V = \mu(Z)$ , where  $Z \in \text{CLC}(Y, \Delta_Y)$  is such that  $\mu(Z) \cap S \neq \emptyset$ .

But, if  $Z$  is such, then  $\mu(Z) \not\subseteq \mu(E)$  for all prime divisor  $E$  on  $Y$  such that  $\mu(E) \cap S = \emptyset$ , so that  $Z \not\subseteq E$  if  $\mu(E) \cap S = \emptyset$ .

Now, if  $F$  is a prime divisor on a normal variety  $Y'$  such that  $h : Y' \rightarrow Y$  is birational and  $h(F) = Z$ , then

$$\begin{aligned} a(F, Y, \Delta_Y) &= a(F, Y, - \sum a(\mu_*^{-1}(D_i), X, \Delta) \mu_*^{-1}(D_i) - \sum_{E \text{ exc.}} a(E, X, \Delta) E) = \\ &= a(F, Y, - \sum_{D_i \cap S \neq \emptyset} a(\mu_*^{-1}(D_i), X, \Delta) \mu_*^{-1}(D_i) - \sum_{\substack{E \text{ exc.} \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta) E) + \\ &\quad + \sum_{\mu(E) \cap S = \emptyset} a(E, X, \Delta) \text{ord}_F(h^*(E)). \end{aligned}$$

But, if  $\mu(E) \cap S = \emptyset$ , then we have seen that  $Z = h(F) \not\subseteq E$ , that is  $\text{ord}_Z(E) = 0$ , so that  $\text{ord}_F(h^*(E)) = 0$ . Thus if we define

$$\Delta'_Y := - \sum_{D_i \cap S \neq \emptyset} a(\mu_*^{-1}(D_i), X, \Delta) \mu_*^{-1} D_i - \sum_{\substack{E \text{ exc.} \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta) E,$$

then

$$\begin{aligned} a(F, Y, \Delta_Y) &= a(F, Y, - \sum_{D_i \cap S \neq \emptyset} a(\mu_*^{-1}(D_i), X, \Delta) \mu_*^{-1} D_i - \sum_{\substack{E \text{ exc.} \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta) E) = \\ &= a(F, Y, \Delta'_Y). \end{aligned}$$

Hence  $Z \in \text{CLC}(Y, \Delta'_Y)$ , so that  $V$  is the image of an element in  $\text{CLC}(Y, \Delta'_Y)$ .

But  $(Y, \Delta'_Y)$  is an LC pair:

In fact, by S-DLTness of  $(X, \Delta)$ , we have that  $-a(\mu_*^{-1}(D_i), X, \Delta) = a_i \leq 1$  if  $D_i \cap S \neq \emptyset$  and, on the other hand, we have proved that if  $E$  is  $\mu$ -exceptional and  $\mu(E) \cap S \neq \emptyset$ , then  $a(E, X, \Delta) > -1$ .

Therefore  $\text{CLC}(Y, \Delta'_Y)$  is finite, so that  $\text{CLC}(X, \Delta, S)$  is finite as well, because every element in  $\text{CLC}(X, \Delta, S)$  is the image of an element in  $\text{CLC}(Y, \Delta'_Y)$ .  $\square$

**Lemma 6.5.** *Let  $(X, \Delta)$  be a pair, let  $S \subseteq X$  be a closed subset and let  $\Delta' \in \text{Div}_{\mathbb{Q}}(X)$  be an effective divisor.*

*If  $(X, \Delta)$  is S-DLT and,  $\text{Supp}(\Delta') \not\supseteq V$ , for every  $V \in \text{CLC}(X, \Delta, S)$ , then there exists a rational number  $\lambda_0 > 0$  such that for every  $\lambda \in \mathbb{Q} \cap [0, \lambda_0]$ , we have that  $(X, \Delta + \lambda \Delta')$  is S-DLT and  $\text{CLC}(X, \Delta + \lambda \Delta', S) = \text{CLC}(X, \Delta, S)$ .*

*Proof.* Let us write

$$\Delta = \sum a_i D_i, \quad \Delta' = \sum b_i D_i,$$

where the  $D_i$ 's are distinct prime divisors on  $X$ , the  $a_i$ 's and the  $b_i$ 's are (possibly zero) rational numbers. In particular, as  $\Delta'$  is effective,  $b_i \geq 0$  for all  $i$ . Hence, for every  $\lambda > 0$ ,

$$\Delta + \lambda \Delta' = \sum (a_i + \lambda b_i) D_i.$$

Now let  $\mu : Y \rightarrow X$  be a standard log-resolution of the pair  $(X, \text{Supp}(\Delta) + \text{Supp}(\Delta'))$ . Define, for every  $\lambda \geq 0$ ,

$$\widetilde{\Delta}_\lambda := \mathbf{A}(\Delta + \lambda \Delta')_Y = - \sum_{E \subseteq Y} a(E, X, \Delta + \lambda \Delta') E,$$

so that, the LC centers of  $(X, \Delta + \lambda\Delta')$  are the images of the LC centers of the pair  $(Y, \widetilde{\Delta}_\lambda)$ . We have that

$$\widetilde{\Delta}_\lambda = - \sum_{E \subseteq Y} a(E, X, \Delta) E + \mu^*(\lambda\Delta'),$$

so that, for every prime divisor  $E \subseteq Y$ , we get

$$a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) - \lambda \operatorname{ord}_E(\mu^*(\Delta')).$$

**Claim 1** *There exists  $\lambda' \in \mathbb{Q} \cap (0, 1]$  such that for every rational number  $\lambda \in [0, \lambda']$  we have the following:*

*If  $E \subseteq Y$  is a prime divisor such that  $\mu(E) \cap S \neq \emptyset$ , then*

- (1)  $a(E, X, \Delta) \geq -1$ ;
- (2)
  - $a(E, X, \Delta) > -1 \Rightarrow a(E, X, \Delta + \lambda\Delta') > -1$ ;
  - $a(E, X, \Delta) = -1 \Rightarrow a(E, X, \Delta + \lambda\Delta') = -1$ .

*Proof of claim 1.*

**1)** If  $E \subseteq Y$  is a  $\mu$ -exceptional prime divisor and  $\mu(E) \cap S \neq \emptyset$ , then, by choice of  $\mu$ ,  $\mu(E) \subseteq \operatorname{Sing}(X) \cup \operatorname{NSNC}(\operatorname{Supp}(\Delta) + \operatorname{Supp}(\Delta')) \subseteq \operatorname{Sing}(X) \cup \operatorname{NSNC}(\Delta) \cup \operatorname{Supp}(\Delta')$ . If, by contradiction  $a(E, X, \Delta) < -1$ , then  $\mu(E) \in \operatorname{CLC}(X, \Delta, S)$ .

By  $S$ -DLTness of  $(X, \Delta)$  this implies that  $\mu(E) \not\subseteq \operatorname{Sing}(X) \cup \operatorname{NSNC}(\Delta)$ . But  $\mu(E) \not\subseteq \operatorname{Supp}(\Delta')$  by hypothesis, so that we find a contradiction.

On the other hand if  $E \subseteq Y$  is a prime divisor, it is not  $\mu$ -exceptional and  $\mu(E) \cap S \neq \emptyset$ , then  $E = \mu_*^{-1}(D_i)$  for some prime divisor  $D_i$  on  $X$  such that  $D_i \cap S \neq \emptyset$ . Then, as  $(X, \Delta)$  is  $S$ -DLT, we have that  $a(E, X, \Delta) = -a_i \geq -1$ .

**2)** Let  $A = \{E \subseteq Y : E \text{ is a prime divisor, } a(E, X, \Delta) > -1, \mu(E) \cap S \neq \emptyset, \operatorname{ord}_E \mu^*(\Delta') \neq 0\}$ .

Then we put

$$\lambda' = \begin{cases} \min_{E \in A} \left\{ \frac{1+a(E, X, \Delta)}{\operatorname{ord}_E \mu^*(\Delta')} \right\} & \text{if } A \neq \emptyset \\ 1 & \text{if } A = \emptyset \end{cases}$$

Let  $E$  be a prime divisor on  $Y$  such that  $\mu(E) \cap S \neq \emptyset$  and  $a(E, X, \Delta) > -1$ . If  $0 \leq \lambda < \lambda'$ , then  $a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) - \lambda \operatorname{ord}_E \mu^*(\Delta') > -1$ .

Now suppose  $E \subseteq Y$  is a prime divisor such that  $\mu(E) \cap S \neq \emptyset$  and  $a(E, X, \Delta) = -1$ . Then  $\mu(E) \in \operatorname{CLC}(X, \Delta, S)$ . Hence  $\operatorname{Supp}(\Delta') \not\supseteq \mu(E)$ , that is  $\operatorname{ord}_E(\mu^*(\Delta')) = 0$ . Thus, for every  $\lambda \geq 0$ ,  $a(E, X, \Delta + \lambda\Delta') = a(E, X, \Delta) = -1$ . This proves claim 1.

**Claim 2** *For every  $\lambda \in \mathbb{Q} \cap [0, \lambda']$ , we have that*

$$\operatorname{CLC}(X, \Delta + \lambda\Delta', S) \subseteq \operatorname{CLC}(X, \Delta, S).$$

If the claim holds, then  $\operatorname{CLC}(X, \Delta + \lambda\Delta', S) = \operatorname{CLC}(X, \Delta, S)$  for every  $\lambda \in \mathbb{Q} \cap [0, \lambda']$ , because  $\operatorname{CLC}(X, \Delta + \lambda\Delta', S) \supseteq \operatorname{CLC}(X, \Delta, S)$  by the effectivity of  $\Delta'$ .

Moreover we can deduce the  $S$ -DLTness of  $(X, \Delta + \lambda\Delta')$ , for  $\lambda \in \mathbb{Q} \cap [0, \lambda']$ :

Property 1 holds because if  $V \in \operatorname{CLC}(X, \Delta + \lambda\Delta', S)$ , then  $V \in \operatorname{CLC}(X, \Delta, S)$ . Then, by the hypotheses,  $V \not\subseteq \operatorname{Sing}(X) \cup \operatorname{NSNC}(\Delta) \cup \operatorname{Supp}(\Delta')$ , so that  $V \not\subseteq \operatorname{Sing}(X) \cup \operatorname{NSNC}(\Delta + \lambda\Delta')$ .

Now suppose  $D_i \cap S \neq \emptyset$ . Then  $a_i + \lambda b_i = -a(\mu_*^{-1}(D_i), X, \Delta + \lambda\Delta') \leq 1$ , thanks to claim 1. Thus property 2 is satisfied for  $\lambda \in \mathbb{Q} \cap [0, \lambda']$ .

The lemma follows by choosing  $\lambda_0 \in \mathbb{Q}$  such that  $0 \leq \lambda_0 < \min\{\frac{1}{m_0}, \lambda'\}$ .

*Proof of claim 2.* If  $V \in CLC(X, \Delta + \lambda\Delta', S)$ , then  $V = \mu(W)$ , for some  $W \in CLC(Y, \widetilde{\Delta}_\lambda)$ , and  $\mu(W) \cap S \neq \emptyset$ .

Hence, if  $E \subseteq Y$  is a prime divisor such that  $\mu(E) \cap S = \emptyset$ , then  $W \not\subseteq E$ , that is  $\text{ord}_W(E) = 0$ . This implies that, for every prime divisor  $F$  over  $W$ , for every  $x \in \mathbb{Q}$ ,

$$a(F, Y, \widetilde{\Delta}_\lambda + xE) = a(F, Y, \widetilde{\Delta}_\lambda).$$

Therefore, if we define

$$\widetilde{\Delta}'_\lambda := - \sum_{\mu(E) \cap S \neq \emptyset} a(E, X, \Delta + \lambda\Delta') E,$$

we get that

$$\begin{aligned} W &\in CLC(Y, \widetilde{\Delta}_\lambda + \sum_{\mu(E) \cap S = \emptyset} a(E, X, \Delta + \lambda\Delta') E) = \\ &= CLC(Y, - \sum_{\mu(E) \cap S \neq \emptyset} a(E, X, \Delta + \lambda\Delta') E) = CLC(Y, \widetilde{\Delta}'_\lambda). \end{aligned}$$

Note that the pair  $(Y, \widetilde{\Delta}'_\lambda)$  is LC by claim 1 and all the LC centers of this pair are irreducible components of intersections of prime divisors in the support of  $(\widetilde{\Delta}'_\lambda)^{=1}$ . But, again by claim 1, we get that

$$\text{Supp}((\widetilde{\Delta}'_\lambda)^{=1}) \subseteq \text{Supp}((-\sum a(E, X, \Delta) E)^{=1}) = \text{Supp}((\widetilde{\Delta}_0)^{=1}).$$

Hence  $W \in CLC(Y, (\widetilde{\Delta}_0)^{=1})$ . Moreover, as  $\widetilde{\Delta}_0$  is SNCS,  $W \in CLC(Y, \widetilde{\Delta}_0)$ . This implies that  $V = \mu(W) \in CLC(X, \Delta, S)$ .  $\square$

The following lemma is an improvement of lemma 2.3.

**Lemma 6.6.** *Let  $(X, \Delta)$  be a pair and let  $S \subseteq X$  be a closed subset such that  $(X, \Delta)$  is an  $S$ -DLT pair. Suppose  $L \in \text{Div}_{\mathbb{Q}}(X)$  is big and nef and  $\mathbb{B}_+(L)$  does not contain any element in  $CLC(X, \Delta, S)$ .*

*Then there exists an effective Cartier divisor  $\Gamma$  on  $X$ , and a rational number  $\lambda_0 > 0$  such that  $Bs(|\Gamma|) = \mathbb{B}_+(L)$  and for each  $\lambda \in \mathbb{Q} \cap (0, \lambda_0]$ , we have that*

- (1)  $L - \lambda\Gamma \in \text{Div}_{\mathbb{Q}}(X)$  and is ample;
- (2)  $CLC(X, \Delta + \lambda\Gamma, S) = CLC(X, \Delta, S)$ ;
- (3)  $(X, \Delta + \lambda\Gamma)$  is an  $S$ -DLT pair.

*Proof.* By [ELMNP06, Prop. 1.5] there exists  $H$ , an ample  $\mathbb{Q}$ -divisor on  $X$ , and  $m_0 \in \mathbb{N}$  such that

$$\mathbb{B}_+(L) = \mathbb{B}(L - H) = Bs(|m_0(L - H)|).$$

Let  $\Gamma \in |m_0(L - H)|$  be a general divisor, so that  $Bs(|\Gamma|) = \mathbb{B}_+(L)$ . For all  $\lambda \in \mathbb{Q} \cap (0, 1]$ , we have that

$$L - \lambda\Gamma \sim_{\mathbb{Q}} (1 - \lambda m_0)L + \lambda m_0 H$$

is ample if  $\lambda \leq \frac{1}{m_0}$  because  $L$  is nef and  $H$  is ample.

As  $CLC(X, \Delta, S)$  is a finite set by lemma 6.4, and as, by hypothesis,  $Bs(|\Gamma|) = \mathbb{B}_+(L)$  does not contain any element in  $CLC(X, \Delta, S)$ , we can choose  $\Gamma$  such that

$\text{Supp}(\Gamma)$  does not contain any element of  $CLC(X, \Delta, S)$ , as well. Thus the lemma follows by lemma 6.5.  $\square$

**Lemma 6.7.** *Let  $(X, \Delta)$  be a pair, let  $S \subseteq X$  be a closed subset and let  $N \in \text{Div}_{\mathbb{Q}}(X)$  be effective.*

*If  $(X, \Delta)$  is  $S$ -DLT, then  $(X, \Delta - N)$  is also  $S$ -DLT.*

*Proof.* Trivially,  $(X, \Delta - N)$  satisfies the property 2, because  $N$  is effective.

Let us prove that  $(X, \Delta - N)$  satisfies the property 1:

Suppose there exists  $V \in CLC(X, \Delta - N, S)$ . Then, by hypothesis,  $V \not\subseteq \text{Sing}(X) \cup NSNC(\Delta)$ .

In order to prove the property 1 we have to show that  $V \not\subseteq NSNC(\Delta - N)$ :

**Claim** *There exists a proper birational morphism  $f : Y \rightarrow X$ , and an irreducible divisor  $F$  on  $Y$  such that  $f(F) = V$  and*

$$a(F, X, \Delta) = a(F, X, \Delta - N) = -1.$$

If the claim holds, then, as usual, we have that

$$a(F, X, \Delta) = a(F, X, \Delta - N) - \text{ord}_F(f^*(N)).$$

Then, by the claim,  $\text{ord}_F(f^*(N)) = 0$ , so that  $V = f(F) \not\subseteq \text{Supp}(N)$ .

Hence  $V \not\subseteq \text{Supp}(N) \cup NSNC(\Delta)$ . As  $NSNC(\Delta - N) \subseteq \text{Supp}(N) \cup NSNC(\Delta)$ , we get that  $V \not\subseteq NSNC(\Delta - N)$ , so that property 1 holds. Thus the lemma will be proved once we prove the claim.

*Proof of the claim.* Let  $\mu : X' \rightarrow X$  be a standard log-resolution of the pair  $(X, \text{Supp}(\Delta))$ .

Let  $E_1, \dots, E_k$  be prime divisors on  $X'$  such that, for all  $j \in \{1, \dots, k\}$ , we have

$$a(E_j, X, \Delta) \neq 0 \quad \text{and} \quad \mu(E_j) \supseteq V.$$

Note that the set of the prime divisors on  $X'$  with this properties is nonempty because  $V \in CLC(X, \Delta)$ . Suppose, furthermore, that  $E_1, \dots, E_k$  are the only prime divisors on  $X'$  with both these properties.

Now suppose there exists  $j \in \{1, \dots, k\}$  such that  $E_j$  is  $\mu$ -exceptional.

Then, by definition of standard log-resolution,

$$\mu(E_j) \subseteq \text{Sing}(X) \cup NSNC(\Delta) \implies V \subseteq \text{Sing}(X) \cup NSNC(\Delta).$$

Thus, all the  $E_j$  are non  $\mu$ -exceptional. Moreover, as  $V \cap S \neq \emptyset$ , we have that  $\mu(E_j) \cap S \neq \emptyset$  for all  $j = 1, \dots, k$ . Then

$$a(E_j, X, \Delta) \geq -1 \quad \forall j = 1, \dots, k,$$

thanks to the  $S$ -DLTness of  $(X, \Delta)$ .

Let  $\nu : X'' \rightarrow X'$  be a proper birational morphism such that there exists a prime divisor  $F \subseteq X''$  such that  $\mu(\nu(F)) = V$  and  $a(F, X, \Delta - N) \leq -1$ . We can suppose that  $X''$  is smooth and that the divisor  $\nu_*^{-1}\mu_*^{-1}\Delta + \nu_*^{-1}\text{exc}(\mu) + \text{exc}(\nu)$  is SNCS. As usual, we find that

$$a(F, X, \Delta) = a(F, X', -\sum a(E, X, \Delta)E) =$$



$$= a(F, X', -\sum_{j=1}^k a(E_j, X, \Delta)E_j) + \sum_{E \neq E_j} a(E, X, \Delta) \operatorname{ord}_F \nu^*(E).$$

But, if  $E \subseteq X'$  is a prime divisor such that  $E \neq E_j$  for all  $j = \{1, \dots, k\}$ , then, either  $a(E, X, \Delta) = 0$ , or  $\operatorname{ord}_F \nu^*(E) = 0$ , because of the choice of the  $E_j$ 's. This shows that

$$a(F, X, \Delta) = a(F, X', -\sum_{j=1}^k a(E_j, X, \Delta)E_j).$$

But  $(X', -\sum_{j=1}^k a(E_j, X, \Delta)E_j)$  is an LC pair, because the divisor  $-\sum_{j=1}^k a(E_j, X, \Delta)E_j$  is SNCS and we have seen that  $a(E_j, X, \Delta) \geq -1$  for all  $j \in \{1, \dots, k\}$ .

Therefore

$$-1 \geq a(F, X, \Delta - N) \geq a(F, X, \Delta) = a(F, X', -\sum_{j=1}^k a(E_j, X, \Delta)E_j) \geq -1,$$

so that  $a(F, X, \Delta - N) = a(F, X, \Delta) = -1$ , and the claim is proved by putting  $f := \mu \circ \nu$  and  $Y := X''$ .  $\square$

## 6.2. Main theorems.

**Theorem 6.8.** *Let  $X$  be a normal projective variety and let  $\Delta$  be an effective Weil  $\mathbb{Q}$ -divisor. If  $D$  is a Weil  $\mathbb{Q}$ -divisor such that*

- (1)  *$D$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $D = P + N$ ;*
- (2)  *$(X, \Delta - aN)$  is a  $P$ -DLT pair;*
- (3) *There exist two rational numbers  $t_0$  and  $a$ , with  $a \geq 0$  such that*

$$t_0 P - (K_X + \Delta - aN)$$

*is ample;*

- (4) *For every  $V \in CLC(X, \Delta - aN)$  we have that  $V \not\subseteq \mathbb{B}(P)$ ;*

*then  $P$  is semiample.*

*Proof.* We define  $A := aN - \Delta$ . Then

$$t_0 P + A - K_X = t_0 P - (K_X + \Delta - aN)$$

is ample.

Now if  $(X, -A)$  is KLT then  $k(X, P) \geq 0$  by Shokurov's nonvanishing theorem (see [KM00, theorem 3.4]).

If  $(X, -A)$  is not KLT then  $CLC(X, -A) \neq \emptyset$ , so that,  $\mathbb{B}(P) \neq X$ , because of 4. Hence, again,  $k(X, P) \geq 0$ .

Then, if we denote

$$\mathbb{N}(P) = \{m \in \mathbb{N} \text{ such that } mP \in \operatorname{Div}(X) \text{ and } H^0(X, \mathcal{O}_X(mP)) \neq 0\},$$

we have that  $\mathbb{N}(P) \neq \emptyset$ . Take  $m_1 \in \mathbb{N}$  such that  $m_1 P$  is a Cartier divisor and  $Bs(|m_1 P|) = \mathbb{B}(P)$ , as closed sets.

We suppose, by contradiction, that  $\mathbb{B}(P) \neq \emptyset$ . We will find  $m \in \mathbb{N}(P)$  and a subvariety  $V \subseteq X$  such that, set-theoretically,  $V \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$  but  $V \not\subseteq Bs(|mP|)$ , leading, in such a way, to a contradiction.

Let  $\{D_j\}_{j=1}^k$  be the finite set of the prime divisors appearing in the support of  $A$  or as base components of  $|m_1 P|$ . We write  $A = \sum_{j=1}^k a_j D_j$ , where the  $a_j$  are possibly zero rational numbers.

Now, as  $(X, -A)$  is  $P$ -DLT and  $\mathbb{B}(P)$  does not contain LC centers of the pair  $(X, \Delta - aN)$ , we can apply lemma 6.3, so that we find  $\mu : Y \rightarrow X$ , a log-resolution of the pair  $(X, -A)$  and of the linear series  $|m_1 P|$  such that:

- $\mu$  is a composition of blowings-up of smooth subvarieties of codimension greater than 1.
- $a(E, X, -A) > -1$  for every  $\mu$ -exceptional prime divisor  $E \subseteq Y$  such that  $\mu(E) \cap \mathbb{B}(P) \neq \emptyset$ ;
- $a(E, X, -A) \geq -1$  for every non- $\mu$ -exceptional prime divisor  $E \subseteq Y$  such that  $\mu(E) \cap \mathbb{B}(P) \neq \emptyset$ ;

Let  $\{F_j = \tilde{D}_j\}_{j=1}^k$  be the finite set of the strict transforms of the divisors  $D_j$  and let  $\{F_j = E_j\}_{j=k+1}^l$  be the finite set of the  $\mu$ -exceptional prime divisors on  $Y$ , so that  $\sum_{j=1}^l F_j$  is a SNC divisor. We can write

$$K_Y \equiv \mu^*(K_X - A) + \sum_{j=1}^l b_j F_j,$$

where  $b_j = a(F_j, X, -A)$  for every  $j = 1, \dots, l$ .

Moreover we can consider an integral base point free divisor  $L$  and coefficients  $r_j \in \mathbb{N} \cup \{0\}$  such that  $\mu^*(m_1 P) = L + \sum r_j F_j$  and  $\mu^*|m_1 P| = |L + \sum r_j F_j|$ .

Hence we have that  $Bs(|m_1 P|) = \mu(\bigcup_{r_j \neq 0} F_j)$ , so that we can suppose  $r_j > 0$  for some  $j$  because  $Bs(|m_1 P|) \neq \emptyset$ .

Moreover if  $r_j \neq 0$ , then  $\mu(F_j) \subseteq Bs(|m_1 P|) = \mathbb{B}(P)$ , which implies that  $b_j > -1$ , as, by hypothesis,  $\mathbb{B}(P)$  does not contain any LC center of the pair  $(X, -A)$ .

Now, as  $t_0 P + A - K_X$  is ample and  $\mu$  is a composition of blowings-up of smooth subvarieties of codimension greater than 1, there exist, for all  $j = k+1, \dots, l$ , arbitrarily small, rational numbers  $\delta_j > 0$ , such that

$$\mu^*(t_0 P + A - K_X) - \sum_{j=k+1}^l \delta_j F_j$$

is still ample.

Thanks to the openness of the ample cone there exist also, for each  $j = 1, \dots, k$ , positive rational numbers  $\delta'_j$  such that if  $0 \leq \delta'_j \leq \delta_j$  then

$$\mu^*(t_0 P + A - K_X) - \sum_{j=1}^k \delta'_j F_j - \sum_{j=k+1}^l \delta_j F_j$$

is ample.

Now we define

$$c = \min_{\{j: r_j \neq 0\}} \frac{b_j + 1 - \delta_j}{r_j}.$$

By choosing the  $\delta_j$ 's small enough we can suppose that  $b_j + 1 - \delta_j > 0$  for all  $j$  such that  $b_j > -1$ . Hence  $c > 0$  because  $b_j > -1$  for every  $j$  such that  $r_j \neq 0$ .

Moreover, perturbing slightly the  $\delta_j$ 's if necessary, we can suppose that the minimum is attained on a unique  $j$ , say  $j = j_0$ . Let  $B := F_{j_0}$ . Now we define

- $J_1 = \{j \in \{1, \dots, k\} \text{ such that } b_j > -1\}$ ,
- $J_2 = \{j \in \{1, \dots, k\} \text{ such that } b_j = -1 \text{ and } D_j \cap \mathbb{B}(P) \neq \emptyset\}$ ,
- $J_3 = \{k+1, \dots, l\}$ ,
- $J_4 = \{j \in \{1, \dots, k\} \text{ such that } b_j \leq -1 \text{ and } D_j \cap \mathbb{B}(P) = \emptyset\}$ .

Note that  $J_1 \amalg J_2 \amalg J_3 \amalg J_4 = \{1, \dots, l\}$ , because, by choice of  $\mu$ ,  $b_j \geq -1$  if  $1 \leq j \leq k$  and  $D_j \cap \mathbb{B}(P) \neq \emptyset$ . Moreover  $j_0 \in J_1 \amalg J_3$  as  $b_0 > -1$ , being  $r_0 \neq 0$ . Now let  $s := t_0 + cm_1$ , and let

$$B_m := \mu^*((m - cm_1)P + A - K_X) - \sum_{j \in J_3} \delta_j F_j.$$

Then, if  $0 \leq \delta'_j \leq \delta_j$  for each  $j \in \{1, \dots, k\} = J_1 \amalg J_2 \amalg J_4$ , we have that for every integer  $m \geq s$

$$B_m - \sum_{J_1 \amalg J_2 \amalg J_4} \delta'_j F_j = \mu^*((m - cm_1)P + A - K_X) - \sum_{j=1}^k \delta'_j F_j - \sum_{j=k+1}^l \delta_j F_j$$

is ample, because  $m - cm_1 \geq t_0$  and  $\mu^*(P)$  is nef.

Let us consider now, for each  $j \in J_2$ , rational, arbitrarily small numbers  $\epsilon_j > 0$ , and define

$$A' := \sum_{j \neq j_0} (-cr_j + b_j - \delta_j) F_j, \quad A'' := A' + \sum_{j \in J_2} (\epsilon_j + \delta_j) F_j.$$

As  $r_j = 0$  if  $b_j \leq -1$  we get that

$$A'' = \sum_{(J_1 \amalg J_3 \amalg J_4) \setminus \{j_0\}} (-cr_j + b_j - \delta_j) F_j + \sum_{J_2} (-1 + \epsilon_j) F_j.$$

Now we define, for every  $m \in \mathbb{N}(P)$ , the divisor

$$Q_m := \mu^*(mP) + A'' - B - K_Y.$$

Then

$$\begin{aligned} Q_m &:= \mu^*(mP) + A' - B - K_Y + (A'' - A') \equiv \\ &\equiv \mu^*(mP) + \sum_{j \neq j_0} (-cr_j + b_j - \delta_j) F_j - F_{j_0} - \sum b_j F_j - \mu^*(K_X - A) + (A'' - A') = \\ &= \mu^*((m - cm_1)P + A - K_X) - \sum_{J_1 \amalg J_3 \amalg J_4} \delta_j F_j + cL + \sum_{J_2} \epsilon_j F_j = \\ &= B_m - \sum_{J_1 \amalg J_4} \delta_j F_j + cL + \sum_{J_2} \epsilon_j F_j. \end{aligned}$$

Let  $m_2 = \min\{m \in \mathbb{N}(P) \text{ such that } m \geq s\}$ . Then  $B_{m_2} - \sum_{J_1 \amalg J_4} \delta_j F_j$  is ample, so that  $Q_{m_2}$  is also ample if the  $\epsilon_j$  are small enough because  $L$  is nef. Hence  $Q_m$  is ample for every  $m \in \mathbb{N}(P)$  such that  $m \geq s$ .

Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we find that  $H^1(Y, \mathcal{O}_Y(\mu^*(mP) + \lceil A'' \rceil - B)) = 0$  if  $m \geq s$  and  $m \in \mathbb{N}(P)$ .

This implies that the restriction homomorphism

$$H^0(Y, \mathcal{O}_Y(\mu^*(mP) + \lceil A'' \rceil)) \rightarrow H^0(B, \mathcal{O}_B(\mu^*(mP) + \lceil A'' \rceil))$$

is surjective in this case.

Now we notice that  $\mu^*(m_2P)|_B + A''|_B - K_B = Q_{m_2}|_B$  is an ample  $\mathbb{Q}$ -divisor.

Moreover  $A''|_B$  is SNCS because  $A''$  is such and  $B$  intersects transversally all the  $F_j$ 's with  $j \neq j_0$ .

Hence it suffices to verify that all the coefficients of  $A''|_B$  are greater than  $-1$  to show that the pair  $(B, -A''|_B)$  is KLT:

Note that if  $j \in J_4$ , then  $\mu(F_j) \cap \mathbb{B}(P) = D_j \cap \mathbb{B}(P) = \emptyset$ , so that  $\mu(F_j) \cap \mu(B) = \emptyset$ , as  $\mu(B) \subseteq \mathbb{B}(P)$ . This implies that  $B \cap F_j = \emptyset$ , that is  $F_j|_B = 0$ .

Moreover if  $j \in J_3$  and  $b_j \leq -1$ , then  $\mu(F_j) \cap \mathbb{B}(P) = \emptyset$ , because of the properties of  $\mu$ , so that, as before, we obtain again that  $F_{j|_B} = 0$ .

Thus

$$\begin{aligned} A''_{|_B} &= \sum_{(J_1 \amalg J_3) \setminus \{j_0\}} (-cr_j + b_j - \delta_j) F_{j|_B} + \sum_{J_2} (-1 + \epsilon_j) F_{j|_B} = \\ &= \sum_{J_1 \setminus \{j_0\}} (-cr_j + b_j - \delta_j) F_{j|_B} + \sum_{\substack{J_3 \setminus \{j_0\} \\ b_j > -1}} (-cr_j + b_j - \delta_j) F_{j|_B} + \sum_{J_2} (-1 + \epsilon_j) F_{j|_B}. \end{aligned}$$

In particular we can suppose that  $j \in J_2$  or  $b_j > -1$ . But we have that

- if  $b_j > -1$  and  $r_j = 0$  then  $-cr_j + b_j - \delta_j = b_j - \delta_j > -1$  (by the choice of the  $\delta_j$ 's);
- if  $b_j > -1$ ,  $r_j \neq 0$  and  $j \neq j_0$  then  $-cr_j + b_j - \delta_j > -\frac{b_j+1-\delta_j}{r_j} r_j + b_j - \delta_j = -1$ .
- if  $j \in J_2$  then  $-1 + \epsilon_j > -1$ ;

Therefore the pair  $(B, -A''_{|_B})$  is KLT. This enables us to use Shokurov's nonvanishing theorem ([KM00, theorem 3.4]), so that for every integer  $k > 0$  we can find  $\mu_k \in \mathbb{N}(P)$ , such that  $\mu_k \geq m_2$ ,  $\mu_k \geq a$ ,  $\mu_k$  is a multiple of  $k$  and

$$H^0(B, \mathcal{O}_B(\mu^*(\mu_k P) + \lceil A'' \rceil)) \neq 0.$$

In fact this cohomology group is non zero for every sufficiently large multiple of  $m_2$ . Let  $k_0 \in \mathbb{N}$  be such that  $k_0 P$  and  $k_0 D$  are integral and  $H^0(X, tk_0 P) \simeq H^0(X, tk_0 D)$  for every  $t \in \mathbb{N}$ . Let us define  $m := \mu_{k_0}$ . Then

$$B \not\subseteq Bs(|\mu^*(mP) + \lceil A'' \rceil|) \text{ and } H^0(X, \mathcal{O}_X(mP)) \simeq H^0(X, \mathcal{O}_X(mD)).$$

Now let us write  $\lceil A'' \rceil = A^+ - A^-$ , where  $A^+$  and  $A^-$  are effective divisors without common components. Note that  $\lceil A'' \rceil = \sum_{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\}} \lceil -cr_j + b_j - \delta_j \rceil F_j$ , so that if we put  $x_j := \lceil -cr_j + b_j - \delta_j \rceil$  for every  $j = 1, \dots, l$ , we have that

$$A^+ = \sum_{\substack{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} x_j F_j, \quad A^- = - \sum_{\substack{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\} \\ x_j < 0}} x_j F_j.$$

Note that  $B \not\subseteq \text{Supp}(A^+)$  and  $B \not\subseteq \text{Supp}(A^-)$ , so that in particular  $B \not\subseteq Bs(|\mu^*(mP) + A^+|)$ .

Moreover we have that  $\mu_*(A^+) \leq \lceil aN \rceil \leq mN$ :

In fact

$$\mu_*(A^+) = \sum_{\substack{J_1 \amalg J_3 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} x_j \mu_*(F_j) = \sum_{\substack{J_1 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} x_j D_j.$$

But, if  $j \in J_1 \amalg J_4$  then  $x_j = \lceil -cr_j + a_j - \delta_j \rceil \leq \lceil a_j \rceil$ . Hence

$$\mu_*(A^+) \leq \sum_{\substack{J_1 \amalg J_4 \setminus \{j_0\} \\ x_j > 0}} \lceil a_j \rceil D_j \leq \sum_{j=1}^k \max\{0, \lceil a_j \rceil\} D_j \leq \lceil aN \rceil$$

as  $\lceil aN \rceil \geq \lceil A \rceil = \sum \lceil a_j \rceil D_j$  and  $\lceil aN \rceil \geq 0$ .

From these inequalities it follows that  $h^0(Y, \mathcal{O}_Y(\mu^*(mP) + A^+)) \leq h^0(X, \mathcal{O}_X(mD))$ .

But

$$\begin{aligned} H^0(X, \mathcal{O}_X(mD)) &\simeq H^0(X, \mathcal{O}_X(mP)) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \hookrightarrow \\ &\hookrightarrow H^0(Y, \mathcal{O}_Y(\mu^*(mP) + A^+)), \end{aligned}$$

so that  $H^0(Y, \mathcal{O}_Y(\mu^*(mP))) \simeq H^0(Y, \mathcal{O}_Y(\mu^*(mP) + A^+))$ .

Therefore we see that  $B \not\subseteq Bs(|\mu^*(mP)|)$ , which implies that  $\mu(B) \not\subseteq Bs(|mP|)$ , giving a contradiction.  $\square$

**Theorem 6.9.** *Let  $(X, \Delta)$  be a pair, with  $\Delta$  effective. If  $D \in \text{Div}_{\mathbb{Q}}(X)$  is such that*

- (1)  *$D$  is big;*
- (2)  *$aD - (K_X + \Delta)$  is nef for some  $a \in \mathbb{Q}$ ;*
- (3) *There exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D)$  admits a  $\mathbb{Q}$ -CKM Zariski decomposition  $f^*(D) = P + N$  and*
  - *$(Z, \mathbf{A}(\Delta)_Z - aN)$  is a  $P$ -DLT pair;*
  - *For every  $V \in \text{CLC}(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(P))$  we have that  $V \not\subseteq \mathbb{B}_+(f^*(D))$ ;*

*then  $P$  is semiample.*

*Proof.* We apply lemma 2.6 and we consider  $t_0, D', P', N', \Delta_Z$  as in the lemma, so that in particular  $D' = P' + N'$  is a  $\mathbb{Q}$ -CKM Zariski decomposition and  $t_0P' - (K_Z + \Delta_Z - N')$  is big and nef. Moreover  $P'$  is big and nef,  $\mathbb{B}_+(P') = \mathbb{B}_+(f^*(D))$  and the pair  $(Z, \Delta_Z - N') = (Z, \mathbf{A}(\Delta)_Z - aN)$  is  $P'$ -DLT.

Then, by lemma 6.6, there exists an effective Cartier divisor  $\Gamma$  on  $Z$  and a rational number  $\lambda > 0$  such that  $P' - \lambda\Gamma$  is ample,  $(Z, \Delta_Z + \lambda\Gamma - N')$  is  $P'$ -DLT and  $\text{CLC}(Z, \Delta_Z + \lambda\Gamma - N') = \text{CLC}(Z, \Delta_Z - N') = \text{CLC}(Z, \mathbf{A}(\Delta)_Z - aN)$ .

Thus

$$(1 + t_0)P' - (K_Z + \Delta_Z + \lambda\Gamma - N') = P' - \lambda\Gamma + t_0P' - (K_Z + \Delta_Z - N')$$

is ample, being the sum of an ample and a nef divisor.

Moreover, as  $\mathbb{B}(P') \subseteq \mathbb{B}_+(P') = \mathbb{B}_+(f^*(D))$ , we have that  $\mathbb{B}(P')$  does not contain any element in  $\text{CLC}(Z, \Delta_Z + \lambda\Gamma - N', \mathbb{B}(P'))$ , so that  $\mathbb{B}(P')$  does not contain any LC center of the pair  $(Z, \Delta_Z + \lambda\Gamma - N')$ .

Therefore we can apply theorem 6.8 and we get that  $P$  is semiample.  $\square$

**Corollary 6.10.** *Let  $(X, \Delta)$  be a pair, with  $\Delta$  effective. If  $D \in \text{Div}_{\mathbb{Q}}(X)$  is such that*

- (1)  *$D$  is big;*
- (2) *There exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D) = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition and*
  - *$(Z, \mathbf{A}(\Delta)_Z)$  is a  $f^*(D)$ -DLT pair;*
  - *For every  $V \in \text{CLC}(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(f^*(D)))$  we have that  $V \not\subseteq \mathbb{B}_+(f^*(D))$ ;*

*then there exists  $\beta > 0$  such that if*

$$aD - (K_X + \Delta) \text{ is nef for some rational number } a > -\beta$$

*then  $P$  is semiample.*

*Proof.* Note that  $\text{Supp}(N) \subseteq \mathbb{B}_+(f^*(D))$ . Then by lemma 6.5 we can find  $\beta > 0$  such that if  $0 \geq a > -\beta$ , then the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$  is  $f^*(D)$ -DLT and  $\text{CLC}(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(f^*(D))) = \text{CLC}(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(f^*(D)))$ .

On the other hand if  $a > 0$  then  $(Z, \mathbf{A}(\Delta)_Z - aN)$  is a  $f^*(D)$ -DLT pair by lemma 6.7 and  $\text{CLC}(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(f^*(D))) \subseteq \text{CLC}(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(f^*(D)))$  by effectivity of  $N$ .

Thus, as  $\mathbb{B}(P) \subseteq \mathbb{B}(f^*(D))$ , for every  $a > -\beta$  we have that  $(Z, \mathbf{A}(\Delta)_Z - aN)$  is a  $P$ -DLT pair and  $\mathbb{B}_+(f^*(D))$  does not contain any LC center in  $\text{CLC}(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(P))$ . Therefore we can apply theorem 6.9.  $\square$

**Remark 6.11.** Note that in theorem 6.9 we may change our hypothesis 3 by assuming that there exists a  $\mathbb{Q}$ -CKM Zariski decomposition  $f^*(D) = P + N$  such that

- $(Z, \mathbf{A}(\Delta)_Z - aN)$  is a  $f^*(D)$ -DLT pair;
- $\mathbb{B}_+(f^*(D))$  does not contain any element in  $CLC(Z, \mathbf{A}(\Delta)_Z - aN, \mathbb{B}(f^*(D)))$ .

In fact we have that  $\mathbb{B}(P) \subseteq \mathbb{B}(f^*(D))$ , so that these assumptions imply hypothesis 3 of the theorem.

On the other hand note that in corollary 6.10 we may replace the hypothesis 2 with the assumptions that there exists a  $\mathbb{Q}$ -CKM Zariski decomposition  $f^*(D) = P + N$  such that  $(Z, \mathbf{A}(\Delta)_Z)$  is a  $P$ -DLT pair and  $\mathbb{B}_+(f^*(D))$  does not contain any LC center in  $CLC(Z, \mathbf{A}(\Delta)_Z, \mathbb{B}(P))$ . This follows by the proof of the corollary itself.

**Corollary 6.12.** *Let  $(X, \Delta)$  be a weak log Fano pair. Suppose that*

- $(X, \Delta)$  is a  $-(K_X + \Delta)$ -DLT pair;
- *for every  $V \in CLC(X, \Delta, \mathbb{B}(-(K_X + \Delta)))$  we have that  $V \not\subseteq \mathbb{B}_+(-(K_X + \Delta))$ ;*

*then  $-(K_X + \Delta)$  is semiample.*

## 7. ALTERNATIVE HYPOTHESES

In this section we state a version of corollary 5.3 with more classical “basepoint-free type” hypotheses and we show that the proof is very similar. Note that the same variation can be stated for proposition 3.1, theorem 3.4 and corollary 5.5.

Moreover these “basepoint-free type” hypotheses already appeared in corollary 3.3 and in theorem 4.2.

**Corollary 7.1.** *Let  $(X, \Delta)$  be a pair such that  $\Delta$  is effective.*

*Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be such that*

- (1)  *$aD - (K_X + \Delta)$  is big and nef for some  $a \in \mathbb{Q}$ ;*
- (2) *there exists a projective birational morphism  $f : Z \rightarrow X$  such that  $f^*(D) = P + N$  is a  $\mathbb{Q}$ -CKM Zariski decomposition and*
  - $(Z, \mathbf{A}(\Delta)_Z - aN)$  *is an LC pair;*
  - $\mathbb{B}_+(f^*(aD - (K_X + \Delta)))$  *does not contain any LC center of the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$ ;*
  - $\widetilde{Nklt}(Z, \mathbf{A}(\Delta)_Z - aN) = \emptyset$ , *or  $P|_{\widetilde{Nklt}(Z, \mathbf{A}(\Delta)_Z - aN)}$  is semiample.*

*Then  $P$  is semiample.*

*Proof.* Define  $L := f^*(aD - (K_X + \Delta))$ . Then we can apply lemma 2.3 to the big and nef  $\mathbb{Q}$ -divisor  $L$  and to the pair  $(Z, \mathbf{A}(\Delta)_Z - aN)$  and we find a Cartier divisor  $\Gamma$  and a rational number  $\lambda > 0$  such that  $L - \lambda\Gamma$  is ample,  $(Z, \mathbf{A}(\Delta)_Z - aN + \lambda\Gamma)$  is LC and  $CLC(Z, \mathbf{A}(\Delta)_Z - aN + \lambda\Gamma) = CLC(Z, \mathbf{A}(\Delta)_Z - aN)$ .

Furthermore, we can choose  $\Gamma$  generically in its linear series and we have that  $Bs(|\Gamma|) = \mathbb{B}_+(L)$ . Then, by Bertini’s lemma, we can suppose that, outside  $\mathbb{B}_+(L)$ ,  $\Gamma$  is smooth and it intersects  $\mathbf{A}(\Delta)_Z - aN$  in a simple normal crossing way.

Now we apply lemma 2.6 and we consider  $t_0, D', P', N', \Delta_Z$  as in the lemma. Then  $t_0P' - (K_Z + \Delta_Z + \lambda\Gamma - N') = P + L - \lambda\Gamma$  is ample.

We conclude by applying theorem 5.2 to the pair  $(Z, \Delta_Z + \lambda\Gamma - N') = (Z, \mathbf{A}(\Delta)_Z - aN + \lambda\Gamma)$  and the  $\mathbb{Q}$ -Cartier divisor  $P'$ :

In fact and we can argue as in the proof of corollary 5.3 to show that all the hypotheses of the theorem are verified.  $\square$

## 8. EXAMPLES

**8.1. Basic construction.** The following general construction is due to Hacon and McKernan (see [Laz09, theorem A.6]). The choice of the surface  $S$  is due to Gongyo (see [Gon09, Example 5.2]).

Let  $S$  be the surface obtained by blowing up  $\mathbb{P}^2$  in 9 very general points, so that  $-K_S$  is nef but not semiample. Let  $S \subseteq \mathbb{P}^N$  be a projectively normal embedding. Let  $X_0$  be the cone over  $S$  and let  $\phi : X \rightarrow X_0$  be the blowing-up at the vertex. We have that  $X \simeq \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(-H))$ , where  $H$  is a sufficiently ample divisor on  $S$ . Now we denote by  $\pi : X \rightarrow S$  the natural projection, and by  $E$  the  $\phi$ -exceptional divisor, so that  $E \simeq S$ .

Note that  $-(K_X + E)$  is big and nef. Hence  $(X, E)$  is a weak log Fano DLT pair and  $E$  is the only LC center of  $(X, E)$ ; in particular it is a PLT pair.

Now, by adjunction, we have that  $-(K_X + E)|_E = -K_E$ , whence  $-(K_X + E)$  is not semiample because  $-K_S$  is not semiample.

**8.2. Applications.** In example 8.1 we will show that, with the notation of the previous subsection,  $E \subseteq \mathbb{B}_+(-(K_X + E))$ , but  $E \not\subseteq \mathbb{B}(-(K_X + E))$ .

Then we have that  $(X, E)$  is a PLT (hence DLT) pair such that

- (1)  $-(K_X + E)$  is big and nef;
- (2)  $\mathbb{B}(-(K_X + E))$  does not contain the only LC center of the pair  $(X, E)$ ;
- (3)  $-(K_X + E)$  is not semiample.

In example 8.2 we will construct, for every  $k \in \mathbb{N}$ , a  $\mathbb{Q}$ -divisor  $P$  and a  $\mathbb{Q}$ -divisor  $\Delta$  on  $X$  such that  $(X, \Delta)$  is DLT and the following conditions are satisfied:

- (1)  $P$  is big and nef;
- (2)  $P - (K_X + \Delta)$  is big and nef;
- (3) The pair  $(X, \Delta)$  has  $m \geq k$  LC centers and just one of these is contained in  $\mathbb{B}_+(P)$ ;
- (4)  $P$  is not semiample.

Note that property 3 implies that there is one LC center of  $(X, \Delta)$ , say  $V$ , such that  $P$  remains big when restricted to every LC center in  $CLC(X, \Delta) \setminus \{V\}$ .

These examples show that in many of our theorems, e.g. proposition 3.1, theorem 3.4 and corollary 5.3, we cannot lighten the hypothesis on  $\mathbb{B}_+$ , in the sense that we cannot replace it with the same hypothesis on the stable base locus and we must take into account *all* the LC centers.

Similarly we cannot sharpen the hypothesis of logbigness of  $P$  in Conjecture 1 as well as in theorem 4.2, in theorem 5.4 and in corollary 5.6.

**Example 8.1.** Note that  $E \subseteq \mathbb{B}_+(-(K_X + E))$  because by Nakamaye's theorem we have that

$$\mathbb{B}_+(-(K_X + E)) = \text{Null}(-(K_X + E))$$

$$\text{and } (-(K_X + E)^2 \cdot E) = (-(K_X + E)|_E)^2 = 0.$$

On the other hand we have that  $E \not\subseteq \mathbb{B}(-(K_X + E))$ :

In fact

$$h^0(E, -(K_X + E)|_E) = h^0(\mathbb{P}^2, \mathcal{I}_{\{p_1, \dots, p_9\}}(3)) \neq 0.$$

Thus the surjectivity of the restriction map

$$H^0(X, -(K_X + E)) \rightarrow H^0(E, -(K_X + E)|_E) \neq 0,$$

given by Kawamata-Viehweg vanishing theorem ([Laz04, theorem 4.3.1]), implies that  $E \not\subseteq Bs(|-(K_X + E)|)$ , so that in particular  $E \not\subseteq \mathbb{B}(-(K_X + E))$ .

**Example 8.2.** Let  $A_1, \dots, A_k$  be smooth hyperplane sections on  $X_0$  such that  $v \notin A_i$  for every  $i = 1, \dots, k$  and the ample divisor  $A := \sum A_i$  is SNC. Let

$$P := -(K_X + E) + \phi^*(A).$$

Moreover define  $\Delta := E + \phi^*(A) = E + \phi_*^{-1}(A)$ .

Note that the pair  $(X, \Delta)$  is DLT, because  $X$  is smooth and  $E + \phi_*^{-1}(A)$  is a SNC divisor, and the LC centers of  $(X, \Delta)$  are exactly the irreducible components of finite intersections of prime divisors in the support of  $\Delta$ , namely  $E$  and  $\phi^*(A_i)$  for every  $i \in \{1, \dots, k\}$ . Note also that  $P$  and  $P - (K_X + \Delta)$  are big and nef.

Now we know that there exists  $\epsilon > 0$  such that  $\phi^*(A) - \epsilon E$  is ample. Then we can write

$$P = (-(K_X + E) + \phi^*(A) - \epsilon E) + \epsilon E,$$

where  $-(K_X + E) + \phi^*(A) - \epsilon E$  is ample.

This implies that  $\mathbb{B}_+(P) \subseteq E$ . On the other hand  $\phi^*(A) \cap E = \emptyset$ , so that the only LC center of the pair  $(X, \Delta)$  contained in  $\mathbb{B}_+(P)$  is  $E$ .

Moreover, as  $\phi^*(A)|_E = 0$ , we have that  $P|_E = -(K_X + E)|_E = -K_E$  is not semi-ample, because  $E \simeq S$ . Therefore  $P$  is not semiample.

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